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The University of Aegean
the University of Crete
and
the University of Athens

co-organize the

10ο Πανελλήνιο Συμπόσιο Λογικής (10th Panhellenic Logic Symposium)

Invited Speakers

J.-Y. Beziau (University of Rio de Janeiro, Brazil)
L. Crosilla (University of Leeds, UK)
P. D'Aquino (University of Napoli II)
V. Gregoriades (TU Darmstadt, Germany)
R. Sklinos (University of Lyon 1, France)
M. Soskova (Sofia University, Bulgaria)
N. Tzevelekos (Queen Mary University of London, UK)
X. Vidaux (University of Concepcion, Chile)

Program Chair

Thanases Pheidas

Organizing Committee

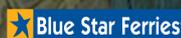
Charalampos Cornaros (Chair)
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Web site

samosweb.aegean.gr/pls10

June 11 - June 15 2015
Samos, Greece

SPONSORS



Preface

The Panhellenic Logic Symposium of 2015 (the tenth PLS) is taking place at the University of the Aegean, on the island of Samos. It is including talks and mini-courses by eight invited well known scientists from around the world. The number of papers submitted to the conference was 42, of which 23 were chosen for presentation, either in oral form or as posters. The selection procedure involved refereeing, mostly by two referees, among the members of the Scientific Committee.

Our effort has been, keeping with the established tradition of the Panhellenic Symposia of past years, to serve the researchers of the wider area of South-Eastern Europe by presenting a wide selection of subjects, to some degree representative of the interests in this part of the world, while adhering to the usual high quality standards of similar events on Mathematics, worldwide. The degree to which we achieved (or not) this aim is to the participants to assess.

Athens, June 2015

Thanases Pheidas
University of Crete
Program Chair, PLS10

The 10th Panhellenic Logic Symposium is taking place on the renowned island of Samos. As Chair of the Organizing Committee, I am grateful to the members of the Steering Committee of the PLS series, for giving me the opportunity to organize this years meeting. The island of Samos is the island of Pythagoras, who not only stated the Pythagorean Theorem, which made him famous as a mathematician, but was also a great philosopher that influenced Plato and Aristotle, as well as the whole of Western philosophy and science. Today, maybe numbers and Mathematics in general do not carry the mysticism they carried in antiquity. But I believe that the thrill caused by a new discovery is the same now as in those days. It is exactly this thrill of discovery that I hope we will share throughout our symposium, during the presentations of results by our select guests. We have the pleasure of hosting eight distinguished invited speakers and over twenty other speakers, who will guide us into the world of modern as well as ancient logic. It would be a serious mistake not to mention the posters, by means of which results will be presented by enthusiastic graduate students, who have come from faraway places. We have done all we could towards making this meeting memorable for all participants, who have gone into great troubles and expenses to come to Samos from all over the planet. To this end, we were lucky to have the help of several sponsors, whose names are mentioned below; if it were not for the local support we have received, the realization of this meeting would have been impossible. The costs of the official dinner, the posters, the proceedings, the coffee breaks and sweets, as well as the tourist guides have been covered almost completely by our sponsors.

Let me finish by thanking aloud the stylobates of this years event: Thanases Pheidas, our Scientific Committee Chair, Costas Dimitracopoulos and Nikos Papaspyrou, my co-organizers, and, of course, Mrs. Nikoleta Tsesmeli, our Secretary, whose organizational experience has compensated for my own inexperience, thus leading to the success of the meeting. I sincerely hope all participants will keep the best memories of the 10th Panhellenic Logic Symposium.

Samos, June 2015

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University of the Aegean
Chair of the Organizing Committee, PLS10

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10th PANHELLENIC LOGIC SYMPOSIUM

Karlovassi, Samos

June 11-15, 2015

PROGRAM

THURSDAY, 11/6/2015

09.00-11.00 Registration

11.00-11.10 Opening

11.10-12.30 Tutorial

- J.-Y. Beziau: Introduction to Universal Logic

12.40-13.30 Invited talk

- R. Sklinos: Some model theory of hyperbolic groups

13.30-16.00 Lunch break

16.00-17.30 Contributed talks

- C. Dimitracopoulos and V. Paschalis: Grades of discernibility
- P. Eleftheriou, A. Hasson and G. Keren: On weakly o-minimal non-valuational structures
- A. Sariev: Definability of jump classes in the local theory of the ω -Turing degrees

17.30-18.00 Coffee break

18.00-19.00 Contributed talks

- A. Soskova, A. Terziyanov and S. Vatev: Generalization of the notion of jump sequence of sets for sequences of structures
- S. Sudoplatov: Generative and pre-generative classes

FRIDAY, 12/6/2015

09.00-10.30 Tutorial

- J.-Y. Beziau: Introduction to Universal Logic

10.30-11.00 Coffee break

11.00-11.50 Invited talk

- L. Crosilla: Understanding predicativity

11.50-12.00 Break

12.00-12.50 Invited talk

- M. Soskova: The enumeration degrees: Local and global structural interactions

12.50-15.30 Lunch break

15.30-17.00 Contributed papers

- C. Cieslinski: Minimalism and the generalization problem
- R. Grandy: Philo v Diodorus revisited: A new conditional in an old setting, or an old conditional in a new setting?
- B. Martin: Dialetheism and dual-valuation logics

17.00-17.30 Coffee break

17.30-19.30 Contributed papers

- E. Rivello: Supervaluation and revision
- V. Sotirov: Leibniz' arithmetized syllogistics: The intensional semantics
- U. Wybraniec-Skardowska: Rejection in traditional and modern logics
- P. Yuste and Á. Garrido: Logic and ontology in ancient India

SATURDAY, 13/6/2015

09.00-10.30 Tutorial

- P. D'Aquino: Real closed fields, models of Peano Arithmetic and recursive saturation

10.30-11.00 Coffee break

11.00-11.50 Invited talk

- V. Gregoriades: Hyperarithmetical incomparability and Kreisel compactness

11.50-12.00 Break

12.00-12.50 Invited talk

- N. Tzevelekos: Automata over infinite alphabets: Investigations in Fresh-Register Automata

12.50-15.00 Lunch break

15.00-17.00 Contributed papers

- F. Dowker, A. Kakas and F. Toni: Argumentation Logic as Anhomomorphic Logic
- D. Georgiev: SQEMA with Universal Modality
- O. Gerasimova and I. Makarov: Total equivalence systems for classes of 3-valued projection logic whose projections equal to the class of linear Boolean functions
- T. Ivanova: Extended contact algebras and internal connectedness

17.00-17.30 Coffee break

17.30-19.30 Contributed papers

- U. Rivieccio: Lukasiewicz public announcement logic
- M. Yanchev: A description logic with transitive and inverse roles, role hierarchies, qualifying number restrictions, and part restrictions
- Anton Zinoviev: Dendral Resolution.

SUNDAY, 14/6/2015

EXCURSION + DINNER

MONDAY, 15/6/2015

09.00-10:30 Tutorial

- P. D'Aquino: Real closed fields, models of Peano Arithmetic and recursive saturation

10.30-11.00 Coffee break

11.00-12.00 Poster session

12.10-13.00 Invited talk

- X. Vidaux: Diophantine undecidability and uniform boundedness of rational points

13.00-13.10 Closing

T.B.A. Syskepsis

10th PANHELLENIC LOGIC SYMPOSIUM

Karlovassi, Samos

June 11-15, 2015

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Part I
Invited Papers and Tutorials

Introduction to Universal Logic

Jean-Yves Béziau

University of Brazil, Rio de Janeiro
University of California, San Diego

Abstract

In a first part I will talk about the general idea of universal logic, its similarity and difference with universal algebra, its relations with philosophy and mathematics, its meaning from the perspective of the history of modern logic. In a second part I will on the one hand explain what kind of mathematical work can be developed in the spirit of universal logic, taking as an example the completeness theorem; on the other hand I will discuss some more philosophical topics about logic and rationality.

References

- [1] J.-Y. Béziau, “13 Questions about universal logic”, *Bulletin of the Section of Logic*, 35 (2006), pp. 133–150.
- [2] J.-Y. Béziau, “The relativity and universality of logic”, *Synthese* – Special Issue Istvan Németi 70th Birthday, March 2014.

Some Model Theory of Hyperbolic Groups

Rizos Sklinos

Université de Lyon 1, France

Abstract

The model theoretic interest about torsion-free hyperbolic groups has been renewed after Sela (2001) answered positively the long standing question of Tarski that concerned the first order theories of non-abelian free groups. Tarski, around 1946, asked whether any two non-abelian free groups share the same common first order theory. Sela's proof can be thought of as the beginning of the interaction between two disciplines of mathematics a priori distant: model theory and geometric group theory. Moreover, as an application of the powerful geometric tools Sela had developed he proved that the first order theory of any torsion-free hyperbolic group is stable. Stability is a very important notion in modern model theory and still dominates its development. It has been discovered by Shelah in the pursuit of his classification program. The stability of torsion-free hyperbolic groups has radically changed the picture concerning stable groups. The only families of groups whose members were already known to be stable were the family of algebraic groups (over algebraically closed fields) and the family of Abelian groups. In this talk I will give the background behind the notion of stability and survey recent results about the first order theories of torsion-free hyperbolic groups as to give our current understanding of their model theory.

Understanding Predicativity

Laura Crosilla*

School of Philosophy, Religion and History of Science
University of Leeds, UK

Abstract

Let's say that a definition is impredicative if it defines an object by quantifying on a totality which includes the object to be defined. A definition is predicative if it is not impredicative.

The subject of predicativity has recently gained prominence in a number of contexts, like, for example, the proof assistant community (see e.g. Coq, Version V8.0). It is a theme that has been running through the constructive tradition, and has found refined expression especially through the work of Per Martin-Löf. Predicativity has also motivated accomplished logical analysis at the hands of prominent logicians in the 1950's and 1960's. Particularly well known is Georg Kreisel's analysis of predicativity, that stimulated fundamental work in proof theory by Solomon Feferman and Kurt Schütte. From its origins, then, predicativity has motivated crucial advances in logic, starting from Russell's very own type theory.

Discussions on predicativity often bring together a number of related but distinct features, making any attempt at delineating the very notion of predicativity extremely complex.

In this talk I shall look at the origins of predicativity, as the analysis of the early discussions on this theme illuminate some of its most fundamental aspects. I shall for example recall that the notion of predicativity originates in the writings of Poincaré and Russell at the beginning of the XX century. Predicativity gave rise to predicativism, which is a philosophical position that takes only predicative reasoning as justified. Herman Weyl expressed a position of this kind in his 1916 book "Das Kontinuum". Like logicism, Hilbert's programme and intuitionism, predicativism is one of a number of influential programmes which arose at the beginning of the past century in an attempt to bring clarity to a fast changing mathematics. The problem of the justification of the new methodology became particularly poignant when paradoxes were discovered in Cantor's and Frege's set theories in the early 20th century. In fact, the paradoxes were the most direct impetus for the very rich discussions between Poincaré and Russell, within which the concept of predicativity was forged. Russell introduced ramified type theory as a way of complying with predicativity. According to an influential analysis of Russell's type theory by Ramsey, further supported by Gödel (1944), predicativity seems too strong a restriction if only the paradoxes are the fundamental motivation for predicativism. In fact, Gödel suggests that a "constructivistic" philosophical position motivates a turn to predicativism, and defends an opposite (platonistic) conception which seems to fully justify impredicativity.

A question naturally arises on the place that predicativity and predicativism may have today, when the threat of the paradoxes is not perceived in equally sinister terms. My favourite approach, stemming from the constructive tradition, sees predicativity as related to our way of understanding mathematical notions. The suggestion then is that a notion of predicativity could help delineate a portion of the mathematical practice that is amenable to a more thorough form of understanding.

*Author supported by ERC grant "The Nature of Representation", <https://natureofrepresentation.wordpress.com/>

The Enumeration Degrees: Local and Global Structural Interactions

Mariya Soskova

Sofia University, Bulgaria

Abstract

We will discuss a hierarchy of reducibilities making the notion of relative definability precise and their algebraic presentations as degree structures. We will concentrate on the middle of this hierarchy: the Turing degrees and the enumeration degrees, which still hold many unsolved mysteries. We will discuss various definability results in the structure of the enumeration degrees, in particular the definability of the total enumeration degrees. The total enumeration degrees are a substructure of the enumeration degree, which is isomorphic to the Turing degrees. These results will be needed in order to exhibit an interesting interaction between the automorphism group of enumeration degrees and the automorphism group of the countable local structure of the computably enumerable degrees.

Real Closed Fields, Models of Peano Arithmetic and Recursive Saturation

Paola D'Aquino

Seconda Università degli Studi di Napoli, Italy

Abstract

I will present a series of results connecting real closed fields and models of Peano Arithmetic and its fragments. This is done via integer parts of real closed fields. Valuation theoretic properties will characterize recursively saturated real closed fields.

Hyperarithmetical Incomparability and Kreisel Compactness

Vassilios Gregoriades

Technische Universität Darmstadt, Germany

Abstract

The hyperarithmetical functions, and consequently the hyperdegrees, arise naturally from problems in effective descriptive set theory. After giving a brief introduction to the basic notions, we present the technique of Kreisel compactness in the area. The latter is a promising approach, which underlines the connection of the hyperarithmetical reducibility with the Turing one. Using this technique we give results about hyperdegrees in some certain classes of Π_1^0 (effectively closed) subsets of the Baire space.

Automata over Infinite Alphabets: Investigations in Fresh-Register Automata

Nikos Tzevelekos

Queen Mary University of London, UK

Abstract

While automata theory has traditionally been based on the model of an abstract machine operating on a finite alphabet of input symbols, this assumption can be unsatisfactory for a range applications where value domains cannot be reasonably bounded: from XML model-checking, to process modelling and the verification of name-generating code. This has recently lead to a surge of interest in automata over infinite alphabets, thus opening up a world of open problems and new results. The talk will look into these exciting developments with a specific focus on latest work driven by applications in modelling and verifying name-generating programs.

Diophantine Undecidability and Uniform Boundedness of Rational Points*

Xavier Vidaux

University of Concepcion, Chile

Abstract

We prove that under the Caporaso-Harris-Mazur conjecture, given an arbitrary integer valued polynomial $F(t)$ of degree at least 2, multiplication is positive existentially definable in the additive group of the integers together with 1 and a unary predicate for the range of $F(t)$. In particular, under the same conjecture, there is no algorithm to decide whether a system of first degree polynomial equations with integral coefficients has or not a solution with some prescribed variables in the range of $F(t)$.

This is joint work with Hector Pasten, partially supported by an Ontario Graduate Scholarship (at Queen's University) and a Benjamin Peirce Fellowship (at Harvard University).

*This work was partially supported by the Chilean Fondecyt research project 1130134., e-mail: xvidaux@udec.cl

Part II
Contributed Papers

Grades of Discernibility

C. Dimitracopoulos^{1,*} and V. Paschalis²

¹ Department of Philosophy and History of Science, University of Athens
GR-157 71, Zografou, Greece. E-mail: cdimitr@phs.uoa.gr

² Department of Mathematics, University of Athens
GR-157 84, Zografou, Greece. E-mail: vpasxal@math.uoa.gr

Abstract

We study grades of discernibility (for pairs of objects), which depend on the quantifier complexity of the formulas defining properties. Our results reveal a hierarchy of grades of discernibility, which is richer than ones studied before.

1 Introduction

The issue of (in)discernibility of objects has been of great interest in both philosophy and logic at least since the time of Leibniz, who proposed his well-known *Principle of the Identity of Indiscernibles* (see [12]), i.e.

$$\forall P(P(x) \leftrightarrow P(y)) \rightarrow x = y \quad (\text{PII}).$$

In studying discernibility, technical tools have been employed recently, at an increasing pace, see, e.g., the work of J. Ketland ([8], [9]) and J. Ladyman, Ø. Linnebo and R. Pettigrew ([11]); these authors studied grades of discernibility by means of first-order languages, essentially continuing in the direction of W. V. Quine's pioneering work on the notion of *discriminability* (see [13], [14]).

Quine defined the notions of *strong discriminability*, *moderate discriminability* and *weak discriminability* for pairs of objects, proved that they correspond to distinct grades of discriminability and asked whether there are any intermediate grades. The terminology has changed since the time of Quine, as can be seen from the table below, which lists the notions studied by Quine and the corresponding notions studied in [9] and in [11].

Quine	Ketland	Ladyman et al.
strong discriminability	monadic discernibility	absolute discernibility
moderate discriminability	relative discernibility	relative discernibility
weak discriminability	weak discernibility	weak discernibility

We note that Quine also defined and studied the notion of *specifiability* of objects, which corresponds to the contemporary notion of *definability* of objects and is intimately related with the notion of discriminability of pairs of objects. Although J. Ketland and J. Ladyman et al. have made significant progress in the study of discernibility in various settings, namely first-order languages with or without the identity symbol, with or without constant symbols (for all the elements of a domain), Quine's question remains unanswered.

If one examines closely the approach taken so far in the study of grades of discernibility, one sees that no attention has been paid to the quantifier complexity of the formulas (of the object

*Research partially supported by project *Aspects and Prospects of Realism in Philosophy of Science and Mathematics*, a THALIS-UoA program co-financed by Greece and the European Union.

language) that are used to define the various grades of discernibility. Our aim in this paper is to refer briefly to work in progress, following a different approach, namely one taking into account the number (of alternations) of quantifiers in the definitions of discernibility notions; the idea is to highlight the rôle of quantifier alternations in definitions of (first-order) properties, in the hope of discovering grades that have not been examined before. This course of action seems to be quite interesting philosophically, as it is similar to the course taken by H. Gaifman in [4], who classified seven positions in the philosophy of mathematics, including the ones adopted by the main schools in the subject, in reference to the quantifier complexity of seven sentences expressing specific mathematical questions. Furthermore, this move is a rather obvious suggestion from the logico-mathematical point of view, given that quantifier complexity has been (for several decades) a natural measure of the complexity of formulas; we recall that many properties of this measure have been studied extensively, e.g. in Recursion Theory, especially in relation to the so-called *arithmetic hierarchy*, and Model Theory, especially in studying the structure of models of Peano Arithmetic and fragments (see, e.g., [7] and [10]).

2 Results

Following [11], we will restrict our study to grades of discernibility relative to first-order languages only; clearly, a similar study relative to other languages, e.g. second-order ones, is worthwhile, but such an undertaking will have to be left for the future. Unlike [11] though, we will restrict ourselves to first-order languages with the identity symbol; the reason behind this decision is that much more is known about structures for such languages than for languages without this symbol; in particular, definability issues have been studied considerably, concerning specific structures for languages with a symbol for the identity relation, so there is a wealth of knowledge to benefit from. Undoubtedly, the study of languages without identity is worthwhile, but it will have to be left for later.

We will assume familiarity of the reader with basic notions of mathematical logic, in particular of model theory; readers not familiar with notions such as formula, structure, satisfaction of a formula in a structure etc. are advised to consult books like [1] and [6]. Before we define our grades of discernibility, we recall the definition of the quantifier hierarchy of formulas of a first-order language \mathcal{L} .

Definition 1. For any $n \in \mathbb{N}$, we say that a formula φ of \mathcal{L} is

- \exists_0 or \forall_0 , if it contains no quantifiers.
- \exists_{n+1} , if it is of the form $\exists x_1 \dots \exists x_k \psi$, where ψ is a \forall_n formula
- \forall_{n+1} , if it is of the form $\forall x_1 \dots \forall x_k \psi$, where ψ is a \exists_n formula.

Remark 2. An immediate consequence of the so-called *Prenex Normal Form Theorem* is that every formula of \mathcal{L} is logically equivalent to an \exists_n (or \forall_n) formula, for a least natural number n (since we may introduce dummy quantifiers, every \exists_n or \forall_n formula is an \exists_k and a \forall_k formula, for any $k > n$).

Now we can proceed to defining our grades of discernibility and definability.

Definition 3. Let \mathcal{A} be a structure for \mathcal{L} , $a, b \in A$ and $n \in \mathbb{N}$. We say that

- (i) a is definable in \mathcal{A} , denoted by $Def_{\mathcal{A}}(a)$, if there exists a formula $\varphi(x)$ of \mathcal{L} such that $\mathcal{A} \models \varphi(a) \wedge \forall y (y \neq a \rightarrow \neg \varphi(y))$ (1).

- (ii) a is \exists_n -definable in \mathcal{A} , denoted by $Def_{\mathcal{A}}\text{-}\exists_n(a)$, if there exists an \exists_n formula $\varphi(x)$ of \mathcal{L} such that (1) holds.
- (iii) a and b are absolutely discernible in \mathcal{A} , denoted by $Abs_{\mathcal{A}}(a,b)$, if there exists a formula $\varphi(x)$ of \mathcal{L} such that $\mathcal{A} \models \varphi(a) \wedge \neg\varphi(b)$ (2).
- (iv) a and b are relatively discernible in \mathcal{A} , denoted by $Rel_{\mathcal{A}}(a,b)$, if there exists a formula $\varphi(x,y)$ of \mathcal{L} such that $\mathcal{A} \models \varphi(a,b) \wedge \neg\varphi(b,a)$ (3).
- (v) a and b are weakly discernible in \mathcal{A} , denoted by $Weak_{\mathcal{A}}(a,b)$, if there exists a formula $\varphi(x,y)$ of \mathcal{L} such that $\mathcal{A} \models \varphi(a,b) \wedge \neg\varphi(a,a)$ (4).
- (vi) a and b are absolutely/relatively/weakly \exists_n -discernible in \mathcal{A} , denoted by $Abs/Rel/Weak_{\mathcal{A}}\text{-}\exists_n(a,b)$, if there exists an \exists_n formula $\varphi(x,y)$ of \mathcal{L} such that (2)/(3)/(4) holds.

Remark 4. Clearly, the notion defined in (ii) is a restricted version of the notion defined in (i), which corresponds to Quine's notion of *specifiability*, while the notions defined in part (vi) are restricted versions of the notions defined in (iii)–(iv), which come from [11].

Following [11] once more, if D and D' are notions of discernibility, we will denote by $D \Rightarrow D'$ the fact that, for every structure \mathcal{A} and every $a, b \in A$, if $D_{\mathcal{A}}(a,b)$, then $D'_{\mathcal{A}}(a,b)$. In the same spirit, we will write $Def\text{-}\exists_n \Rightarrow Abs\text{-}\exists_n$ to mean that, for every structure \mathcal{A} and every $a, b \in A$, if $Def\text{-}\exists_n(a)$, then $Abs\text{-}\exists_n(a,b)$.

Our results are summarized as follows.

Theorem 5.

$$\begin{array}{ccccccc}
Def & \Rightarrow & Abs & \not\Rightarrow & Rel & \not\Rightarrow & Weak \\
\vdots & & \vdots & & \vdots & & \vdots \\
\updownarrow & & \up & & \up & & \up \\
Def\text{-}\exists_n & \Rightarrow & Abs\text{-}\exists_n & \Rightarrow & Rel\text{-}\exists_n & \Rightarrow & Weak\text{-}\exists_n \\
\updownarrow & & \up & & \up & & \up \\
\vdots & & \vdots & & \vdots & & \vdots \\
\updownarrow & & \up & & \up & & \up \\
Def\text{-}\exists_1 & \Rightarrow & Abs\text{-}\exists_1 & \Rightarrow & Rel\text{-}\exists_1 & \Rightarrow & Weak\text{-}\exists_1 \\
\updownarrow & & \up & & \up & & \up \\
Def\text{-}\exists_0 & \Rightarrow & Abs\text{-}\exists_0 & \Rightarrow & Rel\text{-}\exists_0 & \Rightarrow & Weak\text{-}\exists_0
\end{array}$$

Proof. We give the main ideas behind the proofs of the implications and the non-implications of the above diagram.

(a) All vertical implications are trivial, using the fact that, for any $n \in \mathbb{N}$, if φ is an \exists_n formula, then φ is an \exists_k formula, for all $k > n$.

(b) For the implications $Abs \Rightarrow Rel \Rightarrow Weak$ and the non-implications $Weak \not\Rightarrow Rel \not\Rightarrow Abs$, the reader can consult the proof of Theorem 5.1 in [11]. Implications of the form $Abs\text{-}\exists_n \Rightarrow Rel\text{-}\exists_n \Rightarrow Weak\text{-}\exists_n$ are proved similarly.

(c) For the implications $Def(-\exists_n) \Rightarrow Abs(-\exists_n)$, it suffices to observe that for any \mathcal{A} , $a, b \in A$, if a is (\exists_n) -definable in \mathcal{A} , then a and b are absolutely discernible in \mathcal{A} , for all $b \in A, b \neq a$. Indeed, if the formula $\varphi(x)$ defines a in \mathcal{A} , it follows immediately that $\mathcal{A} \models \varphi(a) \wedge \neg\varphi(b)$, for all $b \in A, b \neq a$.

(d) The non-implications $\dots Def\text{-}\exists_n \not\Rightarrow Def\text{-}\exists_{n-1} \not\Rightarrow \dots \not\Rightarrow Def\text{-}\exists_1 \not\Rightarrow Def\text{-}\exists_0$ are proved by exploiting known facts (see, e.g., [2], [5] and [7]) about definable elements in models of (subsystems of) Peano Arithmetic; details can be found in [3]. \square

In fact, the picture concerning definability is much more complicated than the diagram above indicates. To see this, observe first that one can define \forall_n -definability of objects (in structures) in a way similar to that presented in part (ii) of Definition 3. Then one can prove the following result.

Theorem 6.

$$\begin{array}{ccccccc}
 & & Def-\exists_1 & \Rightarrow & \dots & \Rightarrow & Def-\exists_n & \Rightarrow & \dots & \Rightarrow & & & \\
 Def-\exists_0 = Def-\forall_0 & \nearrow & & & & & & & & & & & Def \\
 & & \Downarrow & & & & \Downarrow & & & & & & \\
 & & Def-\forall_1 & \Rightarrow & \dots & \Rightarrow & Def-\forall_n & \Rightarrow & \dots & \Rightarrow & & &
 \end{array}$$

and the converse implications do not hold.

Proof. Again we use known results concerning definable elements in models of (subsystems of) Peano arithmetic; for details, see [3]. \square

Clearly, a lot of work remains to be done; in particular, the following natural questions concerning the diagram in Theorem 5 are unsettled:

- (a) do the converse horizontal implications hold?
- (b) do the converse vertical implications in the columns corresponding to *Abs*, *Rel* and *Weak* hold?
- (c) can the columns corresponding to *Abs*, *Rel* and *Weak* be enriched in the way that the column corresponding to *Def* was enriched in Theorem 6?

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On Weakly o-minimal Non-valuational Structures

Pantelis Eleftheriou^{1,*}, Assaf Hasson^{2,†} and Gil Keren²

¹ Zukunftskolleg and Department of Mathematics and Statistics, University of Konstanz
Box 216, 78457 Konstanz, Germany. E-mail: panteleimon.eleftheriou@uni-konstanz.de

² Department of Mathematics, Ben Gurion University of the Negev, Be'er Sheva, Israel
E-mail: hassonas@math.bgu.ac.il, cruvadam@gmail.com

Abstract

For a weakly o-minimal expansion $\mathcal{M} = \langle M, <, +, \dots \rangle$ of an ordered group, we introduce the notion ‘no external limits’ and prove that \mathcal{M} is o-minimal if and only if it has no external limits and admits definable Skolem functions. We then show that all known examples of weakly o-minimal non-valuational expansions of ordered groups have no external limits and thus obtain a large collection of such structures that do not have definable Skolem functions, extending a result from Shaw [8].

1 Extended abstract

A structure $\mathcal{M} = \langle M, <, \dots \rangle$ is called *o-minimal* if every definable subset of M is a finite union of points and intervals [3, 7]. \mathcal{M} is called *weakly o-minimal* if every definable subset of M is a finite union of convex sets [2, 6]. Examples of weakly o-minimal structures are:

- (a) $\mathcal{R} = \langle \mathcal{R}, \mathcal{F} \rangle \setminus \langle \mathcal{R} \rangle$, a non-archimedean real closed field R expanded by its natural valuation ring $Fin(R)$.
- (b) $\mathcal{R} = \langle \mathbb{R}_{>0}, (t, \pi) \rangle$, the field of real algebraic numbers expanded by the convex set $(0, \pi)$.

These are the archetypical examples of two categories of weakly o-minimal structures that can be distinguished by their ‘definable cuts’. A pair (C, D) of non-empty subsets of M is called a *cut in \mathcal{M}* if $C < D$ and $C \cup D = M$. It is called a *definable cut* if C (and D) are definable. If $\mathcal{M} = \langle M, <, +, \dots \rangle$ expands an ordered group, then \mathcal{M} is called *non-valuational* if for every definable cut (C, D) in M , the infimum $\inf\{y - x : x \in C, y \in D\}$ exists in M (and must equal 0). Otherwise, it is called *valuational*. Example (a) above is valuatinal and (b) is non-valuational. We denote by \overline{M} the set of all definable cuts (C, D) in M such that C has no maximum element. Then \overline{M} has a natural order, where $(C, D) < (C', D')$ if and only if $C < C'$, which extends the order $<$ of M , where $a \in M$ is identified with $((-\infty, a), [a, +\infty))$.

In [6], a weak cell decomposition theorem was proved for every weakly o-minimal structure \mathcal{M} . In [9], a strong cell decomposition theorem was shown in case \mathcal{M} is non-valuational, exhibiting its resemblance to o-minimal structures. In the current note, we introduce the notion of having no external limits (Definition 1.1 below) and use it to prove that a large collection of weakly o-minimal non-valuational structures do not admit definable Skolem functions.

Canonical examples of weakly o-minimal non-valuational structures are obtained by considering dense pairs of o-minimal structures, as introduced in [4]: let \mathcal{N} be an o-minimal structure and $\mathcal{M}_1 \prec \mathcal{N}$ a dense elementary substructure. The theory of dense pairs is the theory of the structure $\langle \mathcal{N}, \mathcal{M}_\infty \rangle$ obtained by expanding \mathcal{N} with a unary predicate for the universe M of \mathcal{M}_∞ . By a result of [1] the structure \mathcal{M} induced from $\langle \mathcal{N}, \mathcal{M}_1 \rangle$ on M is weakly o-minimal.

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Moreover, any definable set in \mathcal{M} is of the form $M^n \cap S$ where $S \subseteq N^n$ is \mathcal{N} -definable, [4, Theorem 2]. Thus, any definable cut in \mathcal{M} is of the form $((-\infty, a) \cap M, (a, +\infty))$ for some $a \in N$. Since M is dense in N , we obtain that \mathcal{M} is non-valuational.

We now give the main definition and results of this note.

Definition 1.1. A weakly o-minimal structure \mathcal{M} has no external limits if for every definable $f : (a, b) \rightarrow M$ where $a, b \in M \cup \{\pm\infty\}$ and $\lim_{t \rightarrow a^+} f(t)$ or $\lim_{t \rightarrow a^+} f(t)$ exist in \overline{M} , then that limit exists in M . Otherwise, we say that \mathcal{M} has external limits.

Theorem 1.2. Let $\langle \mathcal{N}, \mathcal{M}_\infty \rangle$ be a dense pair and \mathcal{M} the induced structure on the universe M of \mathcal{M}_∞ . Then every ordered reduct of \mathcal{M} has no external limits.

By [1], an expansion of an o-minimal structure by any number of convex sets is weakly o-minimal.

Theorem 1.3. Let $\mathcal{M}_\infty = \langle \mathcal{M}, <, +, \dots \rangle$ be an o-minimal expansion of an ordered group. Let $\mathcal{M} = \langle \mathcal{M}_\infty, \{\mathcal{C}_i\}_{i \in \mathcal{I}} \rangle$ be an expansion of \mathcal{M}_∞ by a number of convex sets. Assume that \mathcal{M} is non-valuational. Then there is a dense pair $\langle \mathcal{N}, \mathcal{M}_\infty \rangle$ of o-minimal structures such that \mathcal{M} is the induced structure on M .

It follows that the structure \mathcal{M} from the last theorem has no external limits.

Although it is easy to construct a weakly o-minimal non-valuational structure which has external limits, we conjecture the following:

Conjecture 1.4. Let $\mathcal{M} = \langle \mathcal{M}, <, +, \dots \rangle$ be a weakly o-minimal non-valuational expansion of an ordered group. Then \mathcal{M} has no external limits.

Theorem 1.5. Let $\mathcal{M} = \langle \mathcal{M}, <, +, \dots \rangle$ be a weakly o-minimal expansion of an ordered group. Then \mathcal{M} is o-minimal if and only if it has no external limits and admits definable Skolem functions.

Corollary 1.6. The weakly o-minimal structures from Theorems 1.2 and 1.3 do not admit definable Skolem functions.

The reader may wonder if all weakly o-minimal non-valuational expansions of ordered groups can be obtained as (reducts of) the induced structure from a dense pair $\langle \mathcal{N}, \mathcal{M}_\infty \rangle$ on the universe M of \mathcal{M}_∞ . This is not the case, as it can be shown with the structure $\langle \mathbb{Q}, <, +, \{\{(x, y) : x < \alpha y\}\}_{\alpha \in \mathbb{Q}(\pi)} \rangle$.

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Definability of Jump Classes in the Local Theory of the ω -Turing Degrees

Andrey C. Sariev

Sofia University, Sofia, Bulgaria
acsariev@gmail.com

Abstract

In the present paper we continue the studying of the definability in the local substructure $\mathcal{G}_{T,\omega}$ of the ω -Turing degrees, which was started in the work of Ganchev and Sariev, [6]. We show that the class **I** of the intermediate degrees is definable in $\mathcal{G}_{T,\omega}$.

1 The ω -Turing degrees

A major focus of research in Computability theory involves definability issues in degree structures. Considering a degree structure, natural questions arise about the definability of classes of degrees determined by the structure jump operation. The same questions can be transferred to its local substructures as well. As an interesting special case one can ask, which of the jump classes **H_n**, **L_n** and **I**, consisting of the high_n, the low_n degrees, and the intermediate degrees (i.e. these degrees, which are not neither in **H_n**, nor in **L_n** for any n) respectively, are first order definable in a degree structure.

This paper concerns the problem of the first order definability of the class **I** in the local substructure of the ω -Turing degree structure $\mathcal{D}_{T,\omega}$. Unlike the well known structures of the Turing and the enumeration degrees, $\mathcal{D}_{T,\omega}$ is induced by a reducibility on the set \mathcal{S}_ω of the sequences of sets of natural numbers. The studying of the degree structures induced by a such kind of reducibilities has been initiated by Soskov. In the paper [7], he introduces the ω -enumeration reducibility \leq_ω , considering for each sequence of sets of natural numbers its jump-class – the class consisting of the Turing degrees of all sets, that can compute, in an uniform way, an enumeration of the n -th element of the considering sequence in their n -th Turing jump. Having this, define $\mathcal{A} \leq_\omega \mathcal{B}$ iff the jump-class of \mathcal{B} is a subset of this of \mathcal{A} . The ω -enumeration reducibility is a preorder on \mathcal{S}_ω , and hence it gives rise to a degree structure in the usual way, denoted by \mathcal{D}_ω – the structure of the ω -enumeration degrees.

The structure, an object of the current work, $\mathcal{D}_{T,\omega}$, is introduced by Sariev and Ganchev [6] as a ‘‘Turing’’ analogue of \mathcal{D}_ω in the following way. For every sequence $\mathcal{A} \in \mathcal{S}_\omega$, we define its jump-class $J_{\mathcal{A}}$ to be the set $\{\deg_T(X) \mid A_k \leq_T X^{(k)} \text{ uniformly in } k\}$. Then we set $\mathcal{A} \leq_{T,\omega} \mathcal{B} \iff J_{\mathcal{B}} \subseteq J_{\mathcal{A}}$. Clearly $\leq_{T,\omega}$ is a reflexive and transitive relation, and the relation $\equiv_{T,\omega}$ defined by $\mathcal{A} \equiv_{T,\omega} \mathcal{B} \iff \mathcal{A} \leq_{T,\omega} \mathcal{B} \ \& \ \mathcal{B} \leq_{T,\omega} \mathcal{A}$ is an equivalence relation. The equivalence classes under this relation are called ω -Turing degrees. In particular the equivalence class $\deg_{T,\omega}(\mathcal{A}) = \{\mathcal{B} \mid \mathcal{A} \equiv_{T,\omega} \mathcal{B}\}$ is called the ω -Turing degree of \mathcal{A} . The relation \leq defined by $\mathbf{a} \leq \mathbf{b} \iff \exists \mathcal{A} \in \mathbf{a} \exists \mathcal{B} \in \mathbf{b} (\mathcal{A} \leq_{T,\omega} \mathcal{B})$ is a partial order on the set of all ω -Turing degrees $\mathbf{D}_{T,\omega}$. By $\mathcal{D}_{T,\omega}$ we shall denote the structure $(\mathbf{D}_{T,\omega}, \leq)$. The ω -Turing degree $\mathbf{0}_{T,\omega}$ of the sequence $\emptyset_\omega = \{\emptyset\}_{k < \omega}$ is the least element in $\mathcal{D}_{T,\omega}$. Further, the ω -Turing degree of the sequence $\mathcal{A} \oplus \mathcal{B} = \{A_k \oplus B_k\}_{k < \omega}$ is the least upper bound $\mathbf{a} \vee \mathbf{b}$ of the pair of degrees $\mathbf{a} = \deg_{T,\omega}(\mathcal{A})$ and $\mathbf{b} = \deg_{T,\omega}(\mathcal{B})$. Thus $\mathcal{D}_{T,\omega}$ is an upper semi-lattice with least element.

We need the following definition, in order to characterise the ω -Turing reducibility. Given a sequence $\mathcal{A} \in \mathcal{S}_\omega$ we define the *jump-sequence* $\mathcal{P}(\mathcal{A})$ of \mathcal{A} as the sequence $\{P_k(\mathcal{A})\}_{k < \omega}$ such that $P_0(\mathcal{A}) = A_0$ and for each k , $P_{k+1}(\mathcal{A}) = P_k(\mathcal{A})' \oplus A_{k+1}$.

Now, according to [6], $\mathcal{A} \leq_{T,\omega} \mathcal{B} \iff A_n \leq_T P_n(\mathcal{B})$ uniformly in n . From here, one can show that each sequence is ω -Turing equivalent with its jump-sequence, i.e. for all $\mathcal{A} \in \mathcal{S}_\omega$, $\mathcal{A} \equiv_{T,\omega} \mathcal{P}(\mathcal{A})$.

Further, for the sake of convenience, for sequences $\mathcal{A}, \mathcal{B} \in \mathcal{S}_\omega$ we shall write $\mathcal{A} \leq_T \mathcal{B}$ if and only if for each $k < \omega$, $A_k \leq_T B_k$ uniformly in k . So $\mathcal{A} \leq_{T,\omega} \mathcal{B} \iff \mathcal{A} \leq_T \mathcal{P}(\mathcal{B})$.

Following the lines of [6], the ω -Turing jump \mathcal{A}' of $\mathcal{A} \in \mathcal{S}_\omega$ is defined as the sequence $\mathcal{A}' = (P_1(\mathcal{A}), A_2, A_3, \dots, A_k, \dots)$. This operator is defined so that the jump-class $J_{\mathcal{A}'}$ of \mathcal{A}' contains exactly the jumps of the degrees in the jump-class $J_{\mathcal{A}}$ of \mathcal{A} . Note also, that for each k , $P_k(\mathcal{A}') = P_{1+k}(\mathcal{A})$, so $\mathcal{A}' \equiv_\omega \{P_{k+1}(\mathcal{A})\}$. The jump operator is strictly monotone, i.e. $\mathcal{A} \leq_{T,\omega} \mathcal{A}'$ and $\mathcal{A} \leq_{T,\omega} \mathcal{B} \Rightarrow \mathcal{A}' \leq_{T,\omega} \mathcal{B}'$. This allows to define a jump operation on the ω -Turing degrees by setting $\mathbf{a}' = \text{deg}_{T,\omega}(\mathcal{A}')$, where $\mathcal{A} \in \mathbf{a}$ is an arbitrary. Clearly $\mathbf{a} < \mathbf{a}'$ and $\mathbf{a} \leq \mathbf{b} \Rightarrow \mathbf{a}' \leq \mathbf{b}'$. Let us note that $\mathcal{A}^{(n)} = (P_n(\mathcal{A}), A_{n+1}, A_{n+2}, \dots) \equiv_{T,\omega} \{P_{n+k}(\mathcal{A})\}_{k < \omega}$.

Further, in [6] it is shown that for every natural number n if \mathbf{b} is above $\mathbf{a}^{(n)}$, then there is a least ω -Turing degree \mathbf{x} above \mathbf{a} with $\mathbf{x}^{(n)} = \mathbf{b}$. We shall denote this degree by $\mathbf{I}_\mathbf{a}^n(\mathbf{b})$. An explicit representative of $\mathbf{I}_\mathbf{a}^n(\mathbf{b})$ can be given by setting

$$I_{\mathcal{A}}^n(\mathcal{B}) = (A_0, A_1, \dots, A_{n-1}, B_0, B_1, \dots, B_k, \dots), \quad (1)$$

where each $\mathcal{A} \in \mathbf{a}$ and $\mathcal{B} \in \mathbf{b}$ are arbitrary. From here it follows that for every given $\mathbf{a} \in \mathbf{D}_{T,\omega}$ and $n < \omega$, the operation $\mathbf{I}_\mathbf{a}^n$ is monotone. Also its range is a downwards closed subset of the upper cone with least element \mathbf{a} . Detailed proofs of the all abovementioned properties one can find in [6]. It what follows when $\mathbf{a} = \mathbf{0}_{T,\omega}$ we shall just write \mathbf{I}^n instead of $\mathbf{I}_{\mathbf{0}_{T,\omega}}^n$.

Just like the Turing and the enumeration degree structures, each one of \mathcal{D}_ω and $\mathcal{D}_{T,\omega}$ is augmented with a jump operation – a monotone function which preserves the order. This operation gives rise to the local substructures \mathcal{G}_ω (respectively $\mathcal{G}_{T,\omega}$) of the ω -enumeration (respectively, ω -Turing) degrees. Namely, \mathcal{G}_ω ($\mathcal{G}_{T,\omega}$) consists exactly of the degrees between the least element of \mathcal{D}_ω ($\mathcal{D}_{T,\omega}$) and its first jump. There are several interesting results, concerning definability problems in \mathcal{G}_ω and $\mathcal{G}_{T,\omega}$. Amongst them are the first order definability of each of the classes \mathbf{H}_n and \mathbf{L}_n for each n , both in \mathcal{G}_ω ([2]) and $\mathcal{G}_{T,\omega}$ ([6]), as well as the first order definability of \mathbf{I} in \mathcal{G}_ω , [1]. Note that the latter does not hold in the local substructures neither of the Turing degrees, nor of the enumeration degrees. In this paper we solve the definability problem of the class of the intermediate degrees in $\mathcal{G}_{T,\omega}$.

2 The local theory, jump classes and the \mathbf{o}_n degrees

The structure of the degrees lying beneath the first jump of the least element is usually referred to as the local structure of a degree structure. In the case of the ω -Turing degrees we shall denote this structure by $\mathcal{G}_{T,\omega}$. When considering a local structure, one is usually concerned with questions about the definability of some classes of degrees, which have a natural definition either in the context of the global structure (for example the classes of the high and the low degrees) or in the context of the basic objects from which the degrees are built (for example the class of the Turing degrees containing a c.e. set).

Recall that a degree in the local structure is said to be *high_n* for some n iff its n -th jump is as high as possible. Similarly a degree in the local structure is said to be *low_n* for some n iff its n -th jump is as low as possible. More formally, in the case of $\mathcal{G}_{T,\omega}$, a degree $\mathbf{a} \in \mathcal{G}_{T,\omega}$ is *high_n* iff $\mathbf{a}^{(n)} = (\mathbf{0}_{T,\omega}')^{(n)} = \mathbf{0}_{T,\omega}^{(n+1)}$, and is *low_n* iff $\mathbf{a}^{(n)} = \mathbf{0}_{T,\omega}^{(n)}$.

As usual we shall denote by \mathbf{H}_n the collection of all *high_n* degrees, and by \mathbf{L}_n the collection of all *low_n* degrees. Also \mathbf{H} shall denote the union of all the classes \mathbf{H}_n and analogously \mathbf{L} shall

denote the union of all of the classes \mathbf{L}_n . Finally, \mathbf{I} will stay for the collection of the degrees that are neither high_n nor low_n for any n . The degrees in \mathbf{I} shall be referred to as intermediate degrees.

Using the corresponding results for the structure of the Turing degrees, it is easy to see that there exist intermediate degrees and for every natural number n , there are degrees in the local structure of the ω -Turing degrees, that are $\text{high}_{(n+1)}$ (respectively $\text{low}_{(n+1)}$) but are not high_n (respectively low_n).

Sariev and Ganchev [6] give a characterisation of the classes \mathbf{H}_n and \mathbf{L}_n that does not involve directly the jump operation. Let us set \mathbf{o}_n to be the least n -th jump invert of $\mathbf{0}_{T,\omega}^{(n+1)}$, i.e. $\mathbf{o}_n = \mathbf{I}^n(\mathbf{0}_{T,\omega}^{(n+1)})$. Then for arbitrary $\mathbf{x} \in \mathcal{G}_{T,\omega}$,

$$\mathbf{x} \in \mathbf{H}_n \iff \mathbf{o}_n \leq \mathbf{x}, \text{ and } \mathbf{x} \in \mathbf{L}_n \iff \mathbf{x} \wedge \mathbf{o}_n = \mathbf{0}_{T,\omega}. \quad (2)$$

Note that in order to show that the classes \mathbf{H}_n and \mathbf{L}_n are first order definable in $\mathcal{G}_{T,\omega}$ it is sufficient to show this for the degree \mathbf{o}_n . For the definition of the classes $\mathbf{H} = \bigcup \mathbf{H}_n$, $\mathbf{L} = \bigcup \mathbf{L}_n$ and $\mathbf{I} = \mathcal{G}_{T,\omega} \setminus (\mathbf{H} \cup \mathbf{L})$ it is sufficient to show the definability of the set $\mathfrak{D} = \{\mathbf{o}_n \mid n < \omega\}$.

Note that every high_n degree is also $\text{high}_{(n+1)}$, so $\mathbf{o}_{n+1} \leq \mathbf{o}_n$. Since $\mathbf{H}_{n+1} \setminus \mathbf{H}_n \neq \emptyset$, the equality $\mathbf{o}_{n+1} = \mathbf{o}_n$ is impossible, so that $\mathbf{o}'_\omega = \mathbf{o}_0 > \mathbf{o}_1 > \mathbf{o}_2 > \dots > \mathbf{o}_n > \dots$.

In the above mentioned work of Sariev and Ganchev it is introduced the notion of *almost zero* (*a.z.*) degrees. Namely, the degree \mathbf{x} is *a.z.* iff there is a representative $\mathcal{X} \in \mathbf{x}$ such that $(\forall k)[P_k(\mathcal{X}) \equiv_T \emptyset^{(k)}]$. It is clear that the class of the *a.z.* degrees is downward closed. Note also that there are continuum many *a.z.* degrees and hence not all *a.z.* degrees are in $\mathcal{G}_{T,\omega}$. The *a.z.* degrees in $\mathcal{G}_{T,\omega}$ are exactly the degrees bounded by every degree \mathbf{o}_n , i.e.

$$\mathbf{x} \in \mathcal{G}_{T,\omega} \text{ is } a.z. \iff (\forall n < \omega)[\mathbf{x} \leq \mathbf{o}_n]. \quad (3)$$

We finish this section with some observations concerning the minimal¹ ω -Turing degrees. As it has been shown in [6] there are exactly countably many ω -Turing degrees and all of them are bounded by $\mathbf{0}_{T,\omega}'$. This follows from a characterisation of the minimal degrees in $\mathcal{D}_{T,\omega}$ by Sariev and Ganchev, [6]. Namely, an ω -Turing degree is minimal, if and only if it contains a sequence of the form $(\underbrace{\emptyset, \emptyset, \dots, \emptyset}_n, A, \emptyset, \dots, \emptyset, \dots)$, where the Turing degree of A is a minimal

cover of $\mathbf{0}_T^{(n)}$ and $A' \equiv_T \emptyset^{(n+1)}$. Note, that no *a.z.* degree is minimal. Since each *a.z.* degree bounds only *a.z.* degrees, then no *a.z.* degree bounds a minimal degree. In converse, one can easily show that each of the degrees \mathbf{o}_n bounds (countably many) minimal degrees.

3 Minimal covers and definability in $\mathcal{G}_{T,\omega}$

In this last section we shall show how to first order define the set $\mathfrak{D} = \{\mathbf{o}_n \mid n < \omega\}$. This definition is based on an observation concerning the minimal covers in $\mathcal{G}_{T,\omega}$. From the fact that the set $\mathfrak{D} = \{\mathbf{o}_n \mid n < \omega\}$ is first order definable in $\mathcal{G}_{T,\omega}$, by (2), we conclude the definability of the classes \mathbf{H} , \mathbf{L} and \mathbf{I} .

Using reasoning the very same with this used in characterisation of the minimal ω -Turing degrees in [6], one can easily derive the following characterising theorem.

¹A degree \mathbf{m} is said to be minimal in a degree structure \mathcal{D} , if the only degree strictly less than \mathbf{m} is the least element of \mathcal{D} . Also, \mathbf{m} is a minimal cover of \mathbf{a} iff $\mathbf{a} < \mathbf{m}$ and the interval $\mathbf{D}(\mathbf{a}, \mathbf{m})$ is empty.

Theorem 1. Let $\mathbf{a} < \mathbf{o}_n$ and $\mathcal{A} \in \mathbf{a}$. An ω -Turing degree \mathbf{m} is a minimal cover of \mathbf{a} , which is not below \mathbf{o}_n , iff it contains a sequence of the form $\underbrace{(\emptyset, \emptyset, \dots, \emptyset, M, \emptyset, \dots, \emptyset)}_n, P_n(\mathcal{A}), P_{n+1}(\mathcal{A}) \dots$, where $k < n$, and the Turing degree of M is a minimal cover of $\mathbf{0}_T^{(k)}$. Also, if $k < n - 1$ then $M' \equiv_T \emptyset^{(k+1)}$ and if $k = n - 1$ then $M' \leq_T P_n(\mathcal{A})$.

First note that if $\mathbf{a} < \mathbf{o}_n$ and \mathbf{m} is a minimal cover of \mathbf{a} which is not below the degree \mathbf{o}_n , then, by the characterisation of Theorem 1, $\mathbf{m} \vee \mathbf{o}_n$ is a minimal cover of \mathbf{o}_n itself. This property of the degrees \mathbf{o}_n will be our main component of the definition of the class \mathfrak{D} . For convenience, let $\Phi(\mathbf{x})$ denotes

$$\Phi(\mathbf{x}) \Leftrightarrow (\forall \mathbf{a})(\forall \mathbf{m})[\mathbf{a} < \mathbf{x} \ \& \ \mathbf{m} \text{ is a minimal cover of } \mathbf{a} \ \& \ \mathbf{m} \not\leq \mathbf{x} \rightarrow \mathbf{m} \vee \mathbf{x} \text{ is a minimal cover of } \mathbf{x}].$$

Clearly, Φ is equivalent to a first order formula in the language of the partial orders. Note also, that for each $n < \omega$, $\Phi(\mathbf{o}_n)$. Next, one can notice that if $\mathbf{x} \notin \mathfrak{D}$ is such that $\mathbf{x} > \mathbf{o}_n$ for some natural n then \mathbf{x} has not the property Φ . Indeed, suppose that \mathbf{x} is such a degree and fix the least n for which $\mathbf{o}_n < \mathbf{x}$. Then there is a sequence $\mathcal{X} \in \mathbf{x}$ of the form $(X_0, X_1, \dots, X_{n-1}, \emptyset^{(n+1)}, \emptyset^{(n+2)}, \dots)$.

It is known by a result of Lewis [4], that for each degree \mathbf{a} in the local substructure \mathcal{G}_T of the Turing degrees there exists a minimal degree \mathbf{m} cupping \mathbf{a} to degree of the halting set $\mathbf{0}'_T$, i.e. $\mathbf{a} \vee \mathbf{m} = \mathbf{0}'_T$. The idea here is to use the relativisation of the latter fact for the interval $\mathbf{D}_T[\mathbf{0}_T^{(n-1)}, \mathbf{0}_T^{(n)}]$. Indeed, let \mathbf{m} be a Turing degree such that \mathbf{m} is a minimal cover of $\mathbf{0}_T^{(n-1)}$ which cups the (Turing) degree of $P_{n-1}(\mathcal{X})$ to $\mathbf{0}_T^{(n)}$, i.e. $\mathbf{m} \vee \deg_T(P_{n-1}(\mathcal{X})) = \mathbf{0}_T^{(n)}$. Note that $\mathbf{m}' < \mathbf{0}_T^{(n+1)}$. Now let fix a set M of the \mathbf{m} and consider the sequence $\mathcal{M} = (\underbrace{\emptyset, \emptyset, \dots, \emptyset}_{n-1}, M, \emptyset^{(n+1)}, \emptyset^{(n+2)}, \dots)$. Clearly, $\deg_{T,\omega}(\mathcal{M})$ is a minimal cover of \mathbf{o}_n . Since $P_{n-1} \leq_T \emptyset^{(n)}$ and $M \oplus P_{n-1} \equiv_T \emptyset^{(n)}$, then also we have that \mathbf{x} does not bounds $\deg_{T,\omega}(\mathcal{M})$. Noting that $\mathbf{o}_n < \mathbf{x}$, if we suppose that Φ holds for \mathbf{x} , we shall conclude that the degree $\mathbf{m} \vee \mathbf{x} = \deg_{T,\omega}(\underbrace{P_0(\mathcal{X}), P_1(\mathcal{X}), \dots, P_{n-2}(\mathcal{X})}_{n-1}, \emptyset^{(n)}, \emptyset^{(n+1)}, \dots)$, is a minimal cover of \mathbf{x} . It is not

difficult to note, that if the last is true then $\mathbf{0}_T^{(n)}$ must be a minimal cover of $\deg_T(P_{n-1}(\mathcal{X}))$. But the last does not hold (for example, see Lerman [3]). Hence, $\neg\Phi(\mathbf{x})$.

Lemma 2. Let $\mathbf{x} \in \mathcal{G}_{T,\omega}$ be such that there exists $n < \omega$ for which $\mathbf{o}_n < \mathbf{x}$. Then $\mathcal{G}_{T,\omega} \models \Phi(\mathbf{x})$ if and only if $\mathbf{x} \in \mathfrak{D}$.

Alas, the property stated by Φ is not strong enough to define only the degrees from \mathfrak{D} . We need some additional properties of the degrees \mathbf{o}_n . First, let see which are the degrees that are not covered by the above lemma. If \mathbf{x} is such that there is not $n < \omega$ for which $\mathbf{o}_n < \mathbf{x}$, then there only two possibilities:

- for each n , $\mathbf{x} < \mathbf{o}_n$, i.e. \mathbf{x} is a *a.z.* degree;
- there is a greatest n such that $\mathbf{x} < \mathbf{o}_n$ and for each $m > n$, \mathbf{x} and \mathbf{o}_m are incomparable.

In order to separate the degrees from \mathfrak{D} from the *a.z.* degrees we use the following observation. Recall that no *a.z.* degree bounds a minimal degree. On the other hand for each n , \mathbf{o}_n bounds a minimal degree. Thus the property

$$\Xi(\mathbf{x}) \Leftrightarrow (\exists \mathbf{m})[\mathbf{m} \text{ is a minimal degree} \ \& \ \mathbf{m} < \mathbf{x}]$$

separates \mathfrak{D} from the *a.z.* degrees.

Finally, it remains to handle and the last possible case. In order to do so, we need to recall the definition of each of the degrees \mathfrak{o}_n derived in [6]. Namely, for each $n < \omega$, \mathfrak{o}_{n+1} is the greatest degree which is non cuppable to \mathfrak{o}_n ,². Hence, by Lemma 2, for each $n < \omega$, the degree \mathfrak{o}_n is non cuppable to any degree above it which satisfies Φ . This property of the degrees \mathfrak{o}_n we shall use to finish the definition of the set \mathfrak{D} . Consider the formula Ψ defined as

$$\Psi(\mathbf{x}) \equiv \neg(\exists \mathbf{y})(\exists \mathbf{z})[\mathbf{x} < \mathbf{y} \ \& \ \Phi(\mathbf{y}) \ \& \ \mathbf{y} = \mathbf{x} \vee \mathbf{z}].$$

As it was noted above, for each n , $\Psi(\mathfrak{o}_n)$. Suppose that $\mathbf{x} \in \mathcal{G}_{T,\omega}$ is such that there is a greatest n such that $\mathbf{x} < \mathfrak{o}_n$ and for each $m > n$, \mathbf{x} and \mathfrak{o}_m are incomparable. Then there exists a sequence \mathcal{X} in \mathbf{x} , which is in the form $\mathcal{X} = (\underbrace{\emptyset, \emptyset', \dots, \emptyset^{(n-1)}}_n, X_n, X_{n+1}, \dots)$. Note that

$\emptyset^{(n)} \leq_T P_n(\mathcal{X})$. If $\emptyset^{(n)} \equiv_T P_n(\mathcal{X})$, then $\mathbf{x} < \mathfrak{o}_{n+1}$, which contradicts with the choice of n .

This allows us to use a result by Posner and Robinson [5], by which \mathcal{G}_T features no nonzero noncuppable to $\mathbf{0}'_T$ degrees. Indeed, relativisation of the last property for the substructure of the degrees between $\mathbf{0}_T^{(n)}$ and $\mathbf{0}_T^{(n+1)}$ gives us a degree \mathbf{b} in $\mathbf{D}_T(\mathbf{0}_T^{(n)}, \mathbf{0}_T^{(n+1)})$ such that $\mathbf{b} \vee \deg_T(P_n(\mathcal{X})) = \mathbf{0}_T^{(n+1)}$. Fix $B \in \mathbf{b}$. But if $\mathbf{a} = \deg_{T,\omega}(\underbrace{\emptyset, \emptyset', \dots, \emptyset^{(n-1)}}_n, B, \emptyset^{(n+2)}, \emptyset^{(n+3)}, \dots)$,

then $\mathbf{a} < \mathfrak{o}_n$, $\mathbf{a} \vee \mathbf{x} = \mathfrak{o}_n$ and $\Phi(\mathfrak{o}_n)$. Thus $\neg\Psi(\mathbf{x})$.

Combining all together, we have that for each degree $\mathbf{x} \leq \mathbf{0}'_{T,\omega}$, $\mathbf{x} \in \mathfrak{D} \iff \mathcal{G}_{T,\omega} \models \Phi(\mathbf{x}) \ \& \ \Xi(\mathbf{x}) \ \& \ \Psi(\mathbf{x})$. Thus the following theorem holds.

Theorem 3. *The classes \mathbf{H} , \mathbf{L} and \mathbf{I} are first order definable in $\mathcal{G}_{T,\omega}$*

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²Let $\mathbf{a} < \mathbf{d}$. We shall say that \mathbf{a} is cuppable to \mathbf{d} iff there does exist a degree $\mathbf{b} < \mathbf{d}$ such that the least upper bound $\mathbf{a} \vee \mathbf{b}$ of \mathbf{a} and \mathbf{b} is equal to \mathbf{d} .

Generalization of the Notion of Jump Sequence of Sets for Sequences of Structures

Alexandra Soskova, Alexander Terziivanov, and Stefan Vatev *

Faculty of Mathematics and Informatics, Sofia University
 5 James Bourchier Blvd., 1164 Sofia, Bulgaria
 E-mail: {asoskova, aterziivanov, stefanv}@fmi.uni-sofia.bg

Abstract

We study the notion of relatively intrinsically c.e. sets with respect to a sequence of structures. We propose a generalization of the notion of jump sequence of sets to jump sequence of structures and study the relatively intrinsically c.e. sets in this notion.

1 Introduction

Let $\mathfrak{A} = (A; R_1, \dots, R_k)$ be a countable abstract structure. An enumeration f of \mathfrak{A} is a bijection from \mathbb{N} onto A . For an arbitrary set $X \subseteq A^a$ the pullback of X under the enumeration f is denoted by $f^{-1}(X)$ and defined as $\{\langle x_1 \dots x_a \rangle : (f(x_1), \dots, f(x_a)) \in X\}$. The pullback of the structure \mathfrak{A} under f is $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \dots \oplus f^{-1}(R_k)$. We will consider only structures $\mathfrak{A} = (A; R_1, \bar{R}_1, \dots, R_k, \bar{R}_k)$ where equality is among the predicates R_1, \dots, R_k .

Definition 1.1. *A set $R \subseteq A$ is relatively intrinsically c.e. in \mathfrak{A} if and only if $f^{-1}(R)$ is c.e. in $f^{-1}(\mathfrak{A})$ for every enumeration f of \mathfrak{A} .*

Ash, Knight, Manasse, Slaman[1] and independently Chisholm[2] show that the relatively intrinsically c.e. sets in a structure \mathfrak{A} and the sets that are definable in \mathfrak{A} by means of computable infinitary Σ_1^0 formulas coincide.

We will generalize the notion of jump sequence of a sequence of sets which is the main tool in many results and proofs of Soskov such as the jump inversion theorem for the enumeration jump, the regular enumerations, Ash's theorem for abstract structures and ω -enumeration degrees.

Definition 1.2. *(Soskov) Let $\mathcal{X} = \{X_n\}_{n < \omega}$ and $(\forall n)(X_n \subseteq \mathbb{N})$. The jump sequence $\mathcal{P}(\mathcal{X}) = \{\mathcal{P}_n(\mathcal{X})\}_{n < \omega}$ of \mathcal{X} is defined inductively:*

- (i) $\mathcal{P}_0(\mathcal{X}) = X_0$;
- (ii) $\mathcal{P}_{n+1}(\mathcal{X}) = \mathcal{P}_n(\mathcal{X})'_e \oplus X_{n+1}$. Here $\mathcal{P}_n(\mathcal{X})'_e$ is the enumeration jump of $\mathcal{P}_n(\mathcal{X})$.

We generalize the above notion to a sequence of structures in the following way:

Definition 1.3. *Given a sequence of structures $\vec{\mathfrak{A}} = \{\mathfrak{A}_i\}_{i < \omega}$ the n -th polynomial of $\vec{\mathfrak{A}}$ is a structure $\mathcal{P}_n(\vec{\mathfrak{A}})$ defined inductively:*

- (i) $\mathcal{P}_0(\vec{\mathfrak{A}}) = \mathfrak{A}_0$;
- (ii) $\mathcal{P}_{n+1}(\vec{\mathfrak{A}}) = \mathcal{P}_n(\vec{\mathfrak{A}})' \oplus \mathfrak{A}_{n+1}$. Here the jump of a structure and the join of two structures are appropriately defined.

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We denote by $\mathfrak{A}^{(n)}$ the n -th jump of structure \mathfrak{A} defined inductively:

$$\mathfrak{A}^{(0)} = \mathfrak{A}; \quad \mathfrak{A}^{(n+1)} = (\mathfrak{A}^{(n)})'.$$

Definition 1.4. We call two structures \mathfrak{A} and \mathfrak{B} equivalent: $\mathfrak{A} \equiv \mathfrak{B}$ if they have the same relatively intrinsically c.e. subsets of the common part of the domains of \mathfrak{A} and \mathfrak{B} .

Our main result is the following:

Theorem 1.5. For every sequence of structures $\vec{\mathfrak{A}}$, there exists a structure \mathfrak{M} such that for every n we have $\mathcal{P}_n(\vec{\mathfrak{A}}) \equiv \mathfrak{M}^{(n)}$.

2 Preliminaries

2.1 Enumeration and \leq_n Reducibility

We shall assume a fixed Gödel enumeration W_0, \dots, W_a, \dots of the computably enumerable sets. By D_v we shall denote the finite set with canonical code v . Each c.e. set W_a determines an enumeration operator $W_a : \mathcal{P}(\mathbb{N}) \rightarrow \mathcal{P}(\mathbb{N})$, so that for any sets of natural numbers A and B

$$A = W_a(B) \iff (\forall x)(x \in A \iff (\exists v)(\langle x, v \rangle \in W_a \wedge D_v \subseteq B)).$$

The set A is enumeration reducible to B ($A \leq_e B$) if there exists a c.e. set W such that $A = W(B)$. Let $A \equiv_e B \iff A \leq_e B \ \& \ B \leq_e A$. The relation \equiv_e is an equivalence relation and the respective equivalence classes are called enumeration degrees.

For every set A of natural numbers let $A^+ = A \oplus (\mathbb{N} \setminus A)$. Clearly a set B is c.e. in A if and only if $B \leq_e A^+$. A set A is total if $A \equiv_e A^+$.

Given a set A of natural numbers, set $L_A = \{\langle a, x \rangle : x \in W_a(A)\}$ and let the enumeration jump of A be the set L_A^+ . We will denote it by A'_e . One property of the enumeration jump is $(A^+)'_e \equiv_e (A'_e)^+$ uniformly in A . It is obvious that if A is total then $A'_e \equiv_T A'_T$.

Enumeration reducibility is further generalized to a notion of enumeration reducibility of sets to sequences of sets and to a notion of enumeration reducibility of sequences of sets to sequences of sets. The starting point of these generalizations is Selman's Theorem which states that the set X is enumeration reducible to the set Y if for all sets B , Y is c.e. in B implies X is c.e. in B . The following definition in a different notation is given by Ash:

Definition 2.1. Given a set X of natural numbers and a sequence $\mathcal{Y} = \{Y_k\}_{k \in \omega}$ of sets of natural numbers, let $X \leq_n \mathcal{Y}$ if for all sets $Z \subseteq \mathbb{N}$, \mathcal{Y} is c.e. in Z implies X is Σ_{n+1}^0 in Z . Here \mathcal{Y} is c.e. in Z means that $(\forall k)(Y_k \text{ is c.e. in } Z_T^{(k)} \text{ uniformly in } k)$.

Ash presents a characterization of " \leq_n " using computable infinitary propositional sentences. Another characterization in terms of enumeration reducibility is obtained by Soskov and Kovachev:

Theorem 2.2. (Soskov) $X \leq_n \mathcal{Y}$ if and only if $X \leq_e \mathcal{P}_n(\mathcal{Y})$.

Soskov further generalized the notion to a sequence of structures:

Definition 2.3. Let $\vec{\mathfrak{A}}$ is a sequence of structures and the union of their domains is A . For $R \subseteq A$ we say that $R \leq_n \vec{\mathfrak{A}}$ if $f^{-1}(R) \leq_n f^{-1}(\vec{\mathfrak{A}})$ for every enumeration f of $\vec{\mathfrak{A}}$.

Theorem 2.4 (Soskov[4]). For every sequence of structures $\vec{\mathfrak{A}}$, there exists a structure \mathfrak{M} , such that for each n , the relatively intrinsically Σ_{n+1} sets in \mathfrak{M} sets coincide with the sets $R \leq_n \vec{\mathfrak{A}}$.

The structure \mathfrak{M} is the Marker's extension of the sequence of structures \mathfrak{A} as defined below.

First we will define the n -th Marker's extension $\mathfrak{M}_n(R)$ of $R \subseteq A^m$, where A is the union of the domains. Let X_0, X_1, \dots, X_n be new infinite disjoint countable sets - companions to $\mathfrak{M}_n(R)$. Fix bijections:

$$\begin{aligned} h_0 &: R \rightarrow X_0 \\ h_1 &: (A^m \times X_0) \setminus G_{h_0} \rightarrow X_1 \\ &\dots \\ h_n &: (A^m \times X_0 \times X_1 \cdots \times X_{n-1}) \setminus G_{h_{n-1}} \rightarrow X_n \end{aligned}$$

Let $\mathfrak{M}_n(R) = (A \cup X_0 \cup \dots \cup X_n; X_0, X_1, \dots, X_n, G_{h_n})$.

Now for every n construct the n -th Marker's extension $\mathfrak{M}_n(\mathfrak{A}_n)$ of \mathfrak{A}_n by constructing the n -th Marker's extension for all of its predicates $A_n, R_1^n, \dots, R_{m_n}^n$ with disjoint companions and let $\mathfrak{M}_n(\mathfrak{A}_n) = \mathfrak{M}_n(A_n) \cup \mathfrak{M}_n(R_1^n) \cup \dots \cup \mathfrak{M}_n(R_{m_n}^n)$. Finally for the whole sequence of structures set \mathfrak{M} to be $\bigcup_n \mathfrak{M}_n(\mathfrak{A}_n)$ with one additional predicate for A . For further details refer to [4].

2.2 Moschovakis' Extension and the Jump Structure

Let $\mathfrak{A} = (A; R_1, \dots, R_s)$ be a countable structure and let equality be among the predicates R_1, \dots, R_s . Following Moschovakis[3] the least acceptable extension of the structure \mathfrak{A} is defined as follows.

Let 0 be an object which does not belong to A and Π be a pairing operation chosen so that neither 0 nor any element of A is an ordered pair. Let A^* be the least set containing all elements of $A_0 = A \cup \{0\}$ and closed under operation Π .

Let L and R be the decoding functions on A^* satisfying the following conditions:
 $L(0) = R(0) = 0$; $(\forall t \in A)(L(t) = R(t) = 1^*)$; $(\forall s, t \in A^*)(L(\Pi(s, t)) = s \ \& \ R(\Pi(s, t)) = t)$.

We associate an element n^* of A^* with each natural number n by induction:

$$0^* = 0; \quad (n+1)^* = \Pi(0, n^*).$$

The set of all elements n^* defined above will be denoted by N^* .

The pairing function allows us to code finite sequences of elements: let $\Pi_1(t_1) = t_1$, $\Pi_{n+1}(t_1, t_2, \dots, t_{n+1}) = \Pi(t_1, \Pi_n(t_2, \dots, t_{n+1}))$ for every $t_1, t_2, \dots, t_{n+1} \in A^*$.

For each predicate R_i of the structure \mathfrak{A} define the respective predicate R_i^* on A^* by

$$R_i^*(t) \iff (\exists a_1 \in A) \dots (\exists a_{r_i} \in A)(t = \Pi_{r_i}(a_1, \dots, a_{r_i}) \ \& \ R_i(a_1, \dots, a_{r_i})).$$

Definition 2.5. *Moschovakis' extension of \mathfrak{A} is the structure*

$$\mathfrak{A}^* = (A^*; A_0, R_1^*, \dots, R_s^*, G_\Pi, G_L, G_R, =),$$

where G_Π, G_L and G_R are the graphs of Π, L and R respectively.

We will now define the jump of a structure [5]. We define a forcing with conditions all finite mappings of \mathbb{N} into A . For any $e, x \in \mathbb{N}$ and for every finite mapping δ of \mathbb{N} into A , define the forcing relations $\delta \Vdash F_e(x)$ and $\delta \Vdash \neg F_e(x)$ as follows:

$$\delta \Vdash F_e(x) \iff x \in W_e^{\delta^{-1}(\mathfrak{A})}; \quad \delta \Vdash \neg F_e(x) \iff (\forall \tau \supseteq \delta)(\tau \not\Vdash F_e(x)).$$

Where $\delta^{-1}(\mathfrak{A})$ is a finite function that is an initial part of the characteristic function of $f^{-1}(\mathfrak{A})$ for an enumeration $f \supseteq \delta$ of \mathfrak{A} . We also assume that if the oracle is called with an argument outside the domain of δ then the computation $\{e\}^{\delta^{-1}(\mathfrak{A})}(x)$ halts unsuccessfully.

With each finite mapping $\tau \neq \emptyset$ such that $\text{dom}(\tau) = \{x_1 < \dots < x_n\}$ and $\tau(x_i) = s_i, 1 \leq i \leq n$, we associate an element $\tau^* = \Pi_n(\Pi(x_1^*, s_1), \dots, \Pi(x_n^*, s_n))$ of A^* . Let $\tau^* = 0$ if $\tau = \emptyset$.

Define $K_{\mathfrak{A}} = \{\Pi_3(\delta^*, e^*, x^*) \mid (\exists \tau \supseteq \delta)(\tau \Vdash F_e(x)) \ \& \ e^*, x^* \in N^*\}$. The set $K_{\mathfrak{A}}$ is an analogue of the Kleene set K .

Definition 2.6. We define the jump of structure \mathfrak{A} to be

$$\mathfrak{A}' = (A^*, A_0, R_1^*, \dots, R_s^*, G_\Pi, G_L, G_R, =, K_{\mathfrak{A}}).$$

The main property of the jump structure, obtained in [6], is that for all $X \subseteq A$:

Theorem 2.7. X is relatively intrinsically Σ_{n+1} in $\mathfrak{A} \iff X$ is relatively intrinsically Σ_1 in $\mathfrak{A}^{(n)}$.

Let A be a set, $X \subseteq A$ and f, g are enumerations of A . We will denote by $E_X^{f,g}$ the set:

$$E_X^{f,g} = \{\langle x, y \rangle \mid f(x) = g(y) \in X\}.$$

The following two lemmas give the connection between enumerations of a structure and enumerations of its jump structure. They can be proved following [5](Propositions 13 and 15).

Lemma 2.8. Let \mathfrak{A} be a countable structure with domain A . For every enumeration f of \mathfrak{A} there exists an enumeration g of \mathfrak{A}' such that $g^{-1}(\mathfrak{A}') \leq_T (f^{-1}(\mathfrak{A}))'_T$ and $E_A^{f,g}$ is c.e. in $(f^{-1}(\mathfrak{A}))'_T$.

Lemma 2.9. Let \mathfrak{A} be a countable structure with domain A . For every enumeration f of \mathfrak{A}' there exists an enumeration g of \mathfrak{A} such that $(g^{-1}(\mathfrak{A}))'_T \leq_T f^{-1}(\mathfrak{A}')$ and $E_A^{f,g}$ is c.e. in $f^{-1}(\mathfrak{A}')$.

We now define the join of two structures:

Definition 2.10. Let $\mathfrak{A} = (A; R_1, \dots, R_s, =)$ and $\mathfrak{B} = (B; P_1, \dots, P_t, =)$ be countable structures in the languages \mathfrak{L}_1 and \mathfrak{L}_2 . Suppose that $\mathfrak{L}_1 \cap \mathfrak{L}_2 = \{=\}$ and $A \cap B = \emptyset$. Let $\mathfrak{L} = \mathfrak{L}_1 \cup \mathfrak{L}_2 \cup \{A, B\}$ where A and B are unary predicates. Define $\mathfrak{A} \oplus \mathfrak{B} = (A \cup B; R_1, \dots, R_s, P_1, \dots, P_t, A, B, =)$ in language \mathfrak{L} where predicates A and B are true only on the elements of the domain of \mathfrak{A} and \mathfrak{B} respectively.

In order to satisfy this definition we will only consider sequences of structures $\vec{\mathfrak{A}} = \{\mathfrak{A}_i\}_{i \in \omega}$ where the domains of \mathfrak{A}_i and \mathfrak{A}_j don't have common elements for all $i \neq j$.

3 Proof of main result

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_i\}_{i \in \omega}$ be a sequence of structures and the domain of \mathfrak{A}_i is A_i . Denote by $A^{\leq n} = \bigcup_{i=0}^n A_i$. Let f be an enumeration of $A = \bigcup_{i \in \omega} A_i$ then we denote by \mathcal{P}_n^f the following:

$$\mathcal{P}_0^f = f^{-1}(\mathfrak{A}_0); \quad \mathcal{P}_{n+1}^f = (\mathcal{P}_n^f)'_e \oplus f^{-1}(\mathfrak{A}_{n+1}).$$

Note that for the structures we are considering the set \mathcal{P}_n^f is total for all n and f .

First we prove two lemmas which follow from Lemma 2.8 and Lemma 2.9 by induction:

Lemma 3.1. For every enumeration f of $\vec{\mathfrak{A}}$ and every $n \in \mathbb{N}$ there exists an enumeration g_n of $\mathcal{P}_n(\vec{\mathfrak{A}})$, such that $g_n^{-1}(\mathcal{P}_n(\vec{\mathfrak{A}})) \leq_T \mathcal{P}_n^f$ and $E_{A^{\leq n}}^{f, g_n}$ is c.e. in \mathcal{P}_n^f .

Lemma 3.2. Let $n \in \mathbb{N}$. For every enumeration g of $\mathcal{P}_n(\vec{\mathfrak{A}})$ there exist an enumeration f_n of $\vec{\mathfrak{A}}$, such that $\mathcal{P}_n^{f_n} \leq_T g^{-1}(\mathcal{P}_n(\vec{\mathfrak{A}}))$ and $E_{A^{\leq n}}^{g, f_n}$ is c.e. in $g^{-1}(\mathcal{P}_n(\vec{\mathfrak{A}}))$.

Now using these two lemmas above we can prove the following:

Proposition 3.3. Let $n \in \mathbb{N}$ and $X \subseteq A^{\leq n}$. We have the following equivalence:

$$X \text{ is relatively intrinsically c.e. in } \mathcal{P}_n(\vec{\mathfrak{A}}) \iff X \leq_n \vec{\mathfrak{A}}.$$

PROOF: (\Rightarrow) Suppose that X is relatively intrinsically c.e. in $\mathcal{P}_n(\vec{\mathfrak{A}})$ and let f be an enumeration of A . By Theorem 2.2 we should only prove that $f^{-1}(X) \leq_e \mathcal{P}_n^f$. According to Lemma 3.1 for the enumeration f we can find enumeration g_n of $\mathcal{P}_n(\vec{\mathfrak{A}})$ such that $g_n^{-1}(\mathcal{P}_n(\vec{\mathfrak{A}})) \leq_T \mathcal{P}_n^f$ and $E_{A \leq n}^{f, g_n}$ is c.e. in \mathcal{P}_n^f . Now note the equivalence:

$$x \in f^{-1}(X) \iff (\exists y)((x, y) \in E_{A \leq n}^{f, g_n} \ \& \ y \in g_n^{-1}(X)).$$

Because X is relatively intrinsically c.e. in $\mathcal{P}_n(\vec{\mathfrak{A}})$, the set $g_n^{-1}(X)$ is c.e. in $g_n^{-1}(\mathcal{P}_n(\vec{\mathfrak{A}}))$. Also $g_n^{-1}(\mathcal{P}_n(\vec{\mathfrak{A}})) \leq_T \mathcal{P}_n^f$ from the properties of g_n . Then we have that $g_n^{-1}(X)$ is c.e. in \mathcal{P}_n^f . We also know that $E_{A \leq n}^{f, g_n}$ is c.e. in \mathcal{P}_n^f , so we can conclude that $f^{-1}(X)$ is c.e. in \mathcal{P}_n^f . Since \mathcal{P}_n^f is a total set we have that $f^{-1}(X) \leq_e \mathcal{P}_n^f$.

(\Leftarrow) Suppose that $X \leq_n \vec{\mathfrak{A}}$. Let g be an enumeration of $\mathcal{P}_n(\vec{\mathfrak{A}})$. According to Lemma 3.2 for enumeration g there is an enumeration f_n of $\vec{\mathfrak{A}}$, such that $\mathcal{P}_n^f \leq_T g^{-1}(\mathcal{P}_n(\vec{\mathfrak{A}}))$ and $E_{A \leq n}^{g, f_n}$ is c.e. in $g^{-1}(\mathcal{P}_n(\vec{\mathfrak{A}}))$.

Because $X \leq_n \vec{\mathfrak{A}}$ we have that $f_n^{-1}(X) \leq_e \mathcal{P}_n^f$. The set \mathcal{P}_n^f is total so we also have that $f_n^{-1}(X)$ is c.e. in \mathcal{P}_n^f . We also know that $\mathcal{P}_n^f \leq_T g^{-1}(\mathcal{P}_n(\vec{\mathfrak{A}}))$ by the properties of f_n and so the set $f_n^{-1}(X)$ is c.e. in $g^{-1}(\mathcal{P}_n(\vec{\mathfrak{A}}))$. Now note the equivalence:

$$x \in g^{-1}(X) \iff (\exists y)((x, y) \in E_{A \leq n}^{g, f_n} \ \& \ y \in f_n^{-1}(X)).$$

It is obvious that $g^{-1}(X)$ is c.e. in $g^{-1}(\mathcal{P}_n(\vec{\mathfrak{A}}))$. □

We can now prove our main result Theorem 1.5. Note that the structure \mathfrak{M} is the Marker's extension of the sequence $\vec{\mathfrak{A}}$ and the domains of \mathfrak{M} and $\mathcal{P}_n(\vec{\mathfrak{A}})$ depend on the Moskovakis' extension. We shall assume that the common part of the domains is exactly $A \leq n$.

PROOF: [of Theorem 1.5] Let $n \in \mathbb{N}$. By Theorem 2.7 X is relatively intrinsically c.e. in $\mathfrak{M}^{(n)}$ if and only if when X is relatively intrinsically Σ_{n+1} in \mathfrak{M} .

Now by Theorem 2.4 we have that this is equivalent to $X \leq_n \vec{\mathfrak{A}}$.

Lastly using the previous Proposition 3.3 we conclude that X is relatively intrinsically c.e. in $\mathfrak{M}^{(n)}$ if and only if X is relatively intrinsically c.e. in $\mathcal{P}_n(\vec{\mathfrak{A}})$. Which can be written as:

$$\mathfrak{M}^{(n)} \equiv \mathcal{P}_n(\vec{\mathfrak{A}}).$$

□

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Generative and Pre-generative Classes

Sergey V. Sudoplatov*

Sobolev Institute of Mathematics, Novosibirsk State Technical University,
Novosibirsk State University, Novosibirsk, Russia;
Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
E-mail: sudoplat@math.nsc.ru

Abstract

We investigate axioms for generative classes generalizing them from the countable case to the general one and producing generic structures of an arbitrary theory. We introduce the notions of pre-generative class, regular pre-generative class, and non-refinable generative class. Reducing the list of axioms we prove that any regular pre-generative class can be extended to a non-refinable generative class.

1 Introduction

In a series of papers and books, generic, i. e., generative classes as well as their model-theoretic and related applications are studied by many authors (see [?, ?, ?, ?] for references). We continue this investigation considering axioms for generative classes and generalizing them from the countable case to the general one and producing generic structures of an arbitrary theory. We introduce the notions of pre-generative class, regular pre-generative class, and non-refinable generative class. Reducing the list of axioms we prove that any regular pre-generative class can be extended to a non-refinable generative class.

2 Generative classes and generic limits

Below we write X, Y, Z, \dots for finite sets of variables, and denote by A, B, C, \dots finite sets of elements, as well as finite sets in structures, or else the structures with finite universes themselves.

In diagrams, A, B, C, \dots denote finite sets of constant symbols disjoint from the constant symbols in Σ and $\Sigma(A)$ is the vocabulary with the constants from A adjoined. $\Phi(A), \Psi(B), X(C)$ stand for Σ -*diagrams* (of sets A, B, C), that is, *consistent sets of $\Sigma(A)$ -, $\Sigma(B)$ -, $\Sigma(C)$ -sentences*, respectively.

Below we assume that for any considered diagram $\Phi(A)$, if a_1, a_2 are distinct elements in A then $\neg(a_1 \approx a_2) \in \Phi(A)$. This means that if c is a constant symbol in Σ , then there is at most one element $a \in A$ such that $(a \approx c) \in \Phi(A)$.

If $\Phi(A)$ is a diagram and B is a set, we denote by $\Phi(A)|_B$ the set $\{\varphi(\bar{a}) \in \Phi(A) \mid \bar{a} \in B\}$. Similarly, for a language Σ , we denote by $\Phi(A)|_\Sigma$ the restriction of $\Phi(A)$ to the set of formulas in the language Σ .

Definition. We denote by $[\Phi(A)]_B^A$ the diagram $\Phi(B)$ obtained by replacing a subset $A' \subseteq A$ by a set $B' \subseteq B$ of constants disjoint from Σ and with $|A'| = |B'|$, where $A \setminus A' = B \setminus B'$. Similarly we call the consistent set of formulas denoted by $[\Phi(A)]_X^A$ the type $\Phi(X)$ if it is the result of a bijective substitution into $\Phi(A)$ of variables of X for the constants in A . In this case,

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we say that $\Phi(B)$ is a *copy* of $\Phi(A)$ and a *representative* of $\Phi(X)$. We also denote the diagram $\Phi(A)$ by $[\Phi(X)]_A^X$.

Remark. If the vocabulary contains functional symbols then diagrams $\Phi(A)$ containing equalities and inequalities of terms can generate both finite and infinite structures. The same effect is observed for purely predicate vocabularies if it is written in $\Phi(A)$ that the model for $\Phi(A)$ should be infinite. For instance, diagrams containing axioms for finitely axiomatizable theories have this property.

By the definition, for any diagram $\Phi(A)$, each constant symbol in Σ appears in some formula of $\Phi(A)$. Thus, $\Phi(A)$ can be considered as $\Phi(A \cup K)$, where K is the set of constant symbols in Σ .

We now give conditions on a partial ordering of a collection of diagrams which suffice for it to determine a structure. We modify some of the conditions for structures by d to signify they are conditions on diagrams not structures.

Definition. Let Σ be a vocabulary. We say that $(\mathbf{D}_0; \leq)$ (or \mathbf{D}_0) is *generic*, or *generative*, if \mathbf{D}_0 is a class of Σ -diagrams of finite sets so that \mathbf{D}_0 is partially ordered by a binary relation \leq such that \leq is preserved by bijective substitutions, i. e., if $\Phi(A) \leq \Psi(B)$, and $A' \subseteq B'$ such that $[\Phi(A)]_{A'}^A = \Phi(A')$ and $[\Psi(B)]_{B'}^B = \Psi(B')$ are defined, then $[\Phi(A)]_{A'}^A, [\Psi(B)]_{B'}^B$ are in \mathbf{D}_0 and $[\Phi(A)]_{A'}^A \leq [\Psi(B)]_{B'}^B$.¹ Furthermore:

(i) if $\Phi(A) \in \mathbf{D}_0$ then for any quantifier free formula $\varphi(\bar{x})$ and any tuple $\bar{a} \in A$ either $\varphi(\bar{a}) \in \Phi(A)$ or $\neg\varphi(\bar{a}) \in \Phi(A)$;

(ii) if $\Phi \leq \Psi$ then $\Phi \subseteq \Psi$;²

(iii) if $\Phi \leq X, \Psi \in \mathbf{D}_0$, and $\Phi \subseteq \Psi \subseteq X$, then $\Phi \leq \Psi$;

(iv) some diagram $\Phi_0(\emptyset)$ is the least element of the system $(\mathbf{D}_0; \leq)$;

(v) (the *d-amalgamation property*) for any diagrams $\Phi(A), \Psi(B), X(C) \in \mathbf{D}_0$, if there exist injections $f_0: A \rightarrow B$ and $g_0: A \rightarrow C$ with $[\Phi(A)]_{f_0(A)}^A \leq \Psi(B)$ and $[\Phi(A)]_{g_0(A)}^A \leq X(C)$, then there are a diagram $\Theta(D) \in \mathbf{D}_0$ and injections $f_1: B \rightarrow D$ and $g_1: C \rightarrow D$ for which $[\Psi(B)]_{f_1(B)}^B \leq \Theta(D)$, $[X(C)]_{g_1(C)}^C \leq \Theta(D)$ and $f_0 \circ g_1 = g_0 \circ f_1$; the diagram $\Theta(D)$ is called the *amalgam* of $\Psi(B)$ and $X(C)$ over the diagram $\Phi(A)$ and witnessed by the four maps (f_0, g_0, f_1, g_1) ;

(vi) (the *local realizability property*) if $\Phi(A) \in \mathbf{D}_0$ and $\Phi(A) \vdash \exists x \varphi(x)$, then there are a diagram $\Psi(B) \in \mathbf{D}_0$, $\Phi(A) \leq \Psi(B)$, and an element $b \in B$ for which $\Psi(B) \vdash \varphi(b)$;

(vii) (the *d-uniqueness property*) for any diagrams $\Phi(A), \Psi(B) \in \mathbf{D}_0$ if $A \subseteq B$ and the set $\Phi(A) \cup \Psi(B)$ is consistent then $\Phi(A) = \{\varphi(\bar{b}) \in \Psi(B) \mid \bar{b} \in A\}$.

A diagram Φ is called a *strong subdiagram* of a diagram Ψ if $\Phi \leq \Psi$.

A diagram $\Phi(A)$ is said to be (*strongly*) *embeddable* in a diagram $\Psi(B)$ if there is an injection $f: A \rightarrow B$ such that $[\Phi(A)]_{f(A)}^A \subseteq \Psi(B)$ ($[\Phi(A)]_{f(A)}^A \leq \Psi(B)$). The injection f , in this instance, is called a (*strong*) *embedding* of diagram $\Phi(A)$ in diagram $\Psi(B)$ and is denoted by $f: \Phi(A) \rightarrow \Psi(B)$. A diagram $\Phi(A)$ is said to be (*strongly*) *embeddable* in a structure \mathcal{M} if $\Phi(A)$ is (*strongly*) embeddable in some diagram $\Psi(B)$, where $\mathcal{M} \models \Psi(B)$. The corresponding

¹Note that \mathbf{D}_0 is closed under bijective substitutions since \leq is preserved by bijective substitutions and \leq is reflexive.

²Note that $\Phi(A) \leq \Psi(B)$ implies $A \subseteq B$, since if $a \in A$ then $(a \approx a) \in \Phi(A)$, so $\Phi(A) \leq \Psi(B)$ implies $\Phi(A) \subseteq \Psi(B)$ and we have $(a \approx a) \in \Psi(B)$, whence $a \in B$.

embedding $f: \Phi(A) \rightarrow \Psi(B)$, in this case, is called a (*strong*) *embedding* of diagram $\Phi(A)$ in structure \mathcal{M} and is denoted by $f: \Phi(A) \rightarrow \mathcal{M}$.

Let \mathbf{D}_0 be a class of diagrams, \mathbf{P}_0 be a class of structures of some language, and \mathcal{M} be a structure in \mathbf{P}_0 . The class \mathbf{D}_0 is *cofinal* in the structure \mathcal{M} if for each finite set $A \subseteq M$, there are a finite set B , $A \subseteq B \subseteq M$, and a diagram $\Phi(B) \in \mathbf{D}_0$ such that $\mathcal{M} \models \Phi(B)$. The class \mathbf{D}_0 is *cofinal* in \mathbf{P}_0 if \mathbf{D}_0 is cofinal in every structure of \mathbf{P}_0 . We denote by $\mathbf{K}(\mathbf{D}_0)$ the class of all structures \mathcal{M} with the condition that \mathbf{D}_0 is cofinal in \mathcal{M} , and by \mathbf{P} a subclass of $\mathbf{K}(\mathbf{D}_0)$ such that each diagram $\Phi \in \mathbf{D}_0$ is true in some structure in \mathbf{P} .

Now we extend the relation \leq from the generative class $(\mathbf{D}_0; \leq)$ to a class of subsets of structures in the class $\mathbf{K}(\mathbf{D}_0)$.

Let \mathcal{M} be a structure in $\mathbf{K}(\mathbf{D}_0)$, A and B be finite sets in \mathcal{M} with $A \subseteq B$. We call A a *strong subset* of the set B (in the structure \mathcal{M}), and write $A \leq B$, if there exist diagrams $\Phi(A), \Psi(B) \in \mathbf{D}_0$, for which $\Phi(A) \leq \Psi(B)$ and $\mathcal{M} \models \Psi(B)$.

A finite set A is called a *strong subset* of a set $M_0 \subseteq M$ (in the structure \mathcal{M}), where $A \subseteq M_0$, if $A \leq B$ for any finite set B such that $A \subseteq B \subseteq M_0$ and $\Phi(A) \leq \Psi(B)$ for some diagrams $\Phi(A), \Psi(B) \in \mathbf{D}_0$ with $\mathcal{M} \models \Psi(B)$. If A is a strong subset of M_0 then, as above, we write $A \leq M_0$. If $A \leq M$ in \mathcal{M} then we refer to A as a *self-sufficient set* (in \mathcal{M}).

Notice that, by the d -uniqueness property, the diagrams $\Phi(A)$ and $\Psi(B)$ specified in the definition of strong subsets are defined uniquely. A diagram $\Phi(A) \in \mathbf{D}_0$, corresponding to a self-sufficient set A in \mathcal{M} , is said to be a *self-sufficient diagram* (in \mathcal{M}).

Definition. A class $(\mathbf{D}_0; \leq)$ possesses the *joint embedding property* (JEP) if for any diagrams $\Phi(A), \Psi(B) \in \mathbf{D}_0$, there is a diagram $X(C) \in \mathbf{D}_0$ such that $\Phi(A)$ and $\Psi(B)$ are strongly embeddable in $X(C)$.

Clearly, every generative class has JEP since JEP means the d -amalgamation property over the empty set.

Definition. A structure $\mathcal{M} \in \mathbf{P}$ has *finite closures* with respect to the class $(\mathbf{D}_0; \leq)$, or is *finitely generated over Σ* , if any finite set $A \subseteq M$ is contained in some finite self-sufficient set in \mathcal{M} , i. e., there is a finite set B with $A \subseteq B \subseteq M$ and $\Psi(B) \in \mathbf{D}_0$ such that $\mathcal{M} \models \Psi(B)$ and $\Psi(B) \leq X(C)$ for any $X(C) \in \mathbf{D}_0$ with $\mathcal{M} \models X(C)$ and $\Psi(B) \subseteq X(C)$. A class \mathbf{P} has *finite closures* with respect to the class $(\mathbf{D}_0; \leq)$, or is *finitely generated over Σ* , if each structure in \mathbf{P} has finite closures (with respect to $(\mathbf{D}_0; \leq)$).

Clearly, an at most countable structure \mathcal{M} has finite closures with respect to $(\mathbf{D}_0; \leq)$ if and only if $M = \bigcup_{i \in \omega} A_i$ for some self-sufficient sets A_i with $A_i \leq A_{i+1}$, $i \in \omega$.

Note that the finite closure property is defined modulo Σ and does not correlate with the cardinalities of algebraic closures. For instance, if Σ contains infinitely many constant symbols then $\text{acl}(A)$ is always infinite whereas a finite set A can or can not be extended to a self-sufficient set.

Besides, for the finite closures of sets A we consider finite self-sufficient extensions B in a given structure \mathcal{M} with respect to $(\mathbf{D}_0; \leq)$ only and B can be both a universe of a substructure of \mathcal{M} or not. Moreover, it is permitted that corresponding diagrams $\Psi(B)$ can have only finite, finite and infinite, or only infinite models.

Thus, for instance, a finitely axiomatizable theory without finite models and with a generative class $(\mathbf{D}_0; \subseteq)$, containing diagrams for all finite sets and with axioms in diagrams, has identical finite closures whereas each diagram in \mathbf{D}_0 has only infinite models.

Definition. A structure $\mathcal{M} \in \overline{\mathbf{D}}_0$ is $(\mathbf{D}_0; \leq)$ -*generic*, or a *generic limit for the class $(\mathbf{D}_0; \leq)$* and denoted by $\text{glim}(\mathbf{D}_0; \leq)$, if it satisfies the following conditions:

- (a) \mathcal{M} has finite closures with respect to \mathbf{D}_0 ;
 (b) if $A \subseteq M$ is a finite set, $\Phi(A), \Psi(B) \in \mathbf{D}_0$, $\mathcal{M} \models \Phi(A)$ and $\Phi(A) \leq \Psi(B)$, then there exists a set $B' \leq M$ such that $A \subseteq B'$ and $\mathcal{M} \models \Psi(B')$.

Given any generative class $(\mathbf{D}_0; \leq)$, we can embark on a upward directed construction of a $(\mathbf{D}_0; \leq)$ -generic structure \mathcal{M} using d -amalgamations and local realizabilities. Thus the following theorem holds.

Theorem 1. *For any generative class $(\mathbf{D}_0; \leq)$, there exists a $(\mathbf{D}_0; \leq)$ -generic structure.*

Theorem 2. *Every ω -homogeneous structure \mathcal{M} is a $(\mathbf{D}_0; \leq)$ -generic structure for some generative class $(\mathbf{D}_0; \leq)$.*

Proof. Let \mathcal{M} be an ω -homogeneous structure. The required class is the class $(\mathbf{D}_0; \leq)$, where \mathbf{D}_0 consists of all copies of complete diagrams $\Phi(A) = \{\varphi(\bar{a}) \mid \mathcal{M} \models \varphi(\bar{a}) \text{ where } \bar{a} \in A\}$ (with respect to \mathcal{M}) for every finite set $A \subseteq M$, and \leq is the inclusion relation.

Indeed, checking the d -amalgamation property we consider complete diagrams $[\text{tp}_X(\bar{a})]_A^X$, $[\text{tp}_Y(\bar{b})]_B^Y$, $[\text{tp}_Z(\bar{c})]_C^Z$, where $[\text{tp}_X(\bar{a})]_A^X = [\text{tp}_Y(\bar{b})]_B^Y \cap [\text{tp}_Z(\bar{c})]_C^Z$, $\bar{a}, \bar{b}, \bar{c}$ are tuples in \mathcal{M} , $\text{tp}(\bar{a}) = \text{tp}(\bar{b}) \cap \text{tp}(\bar{c})$. Then we take $[\text{tp}_{YZ}(\bar{b}\bar{c})]_{BC}^{YZ}$ as the amalgam of $[\text{tp}_Y(\bar{b})]_B^Y$ and $[\text{tp}_Z(\bar{c})]_C^Z$ over $[\text{tp}_X(\bar{a})]_A^X$. Other properties of generative classes immediately hold. \square

Thus any first-order theory has a generic model and therefore can be represented by it.

3 Pre-generative classes

Consider the following modification of the d -amalgamation property for a class $(\mathbf{D}_0; \leq)$:

(v') for any diagrams $\Phi(A), \Psi(B), X(C) \in \mathbf{D}_0$, if there exist injections $f_0: A \rightarrow B$ and $g_0: A \rightarrow C$ with $[\Phi(A)]_{f_0(A)}^A \leq \Psi(B)$ and $[\Phi(A)]_{g_0(A)}^A \leq X(C)$ such that $B \setminus A = B \setminus f_0(A)$, $C \setminus A = C \setminus g_0(A)$, $(B \setminus A) \cap (C \setminus A) = \emptyset$ and $[\Psi(B)]_{f_0(A) \cup [X(C)]_A^{g_0(A)}}^{f_0(A)} \cup [X(C)]_A^{g_0(A)}$ is consistent, then there are a diagram $\Theta(D) \in \mathbf{D}_0$ and injections $f_1: B \rightarrow D$ and $g_1: C \rightarrow D$ for which $[\Psi(B)]_{f_1(B)}^B \leq \Theta(D)$, $[X(C)]_{g_1(C)}^C \leq \Theta(D)$ and $f_0 \circ f_1 = g_0 \circ g_1$.

Note that replacing the d -amalgamation property in the definition of generative class by (v') it suffices to use only identical embeddings for the construction of $(\mathbf{D}_0; \leq)$ -generic structure.

A generative class $(\mathbf{D}_0; \leq)$ is *self-sufficient* if the following *axiom of self-sufficiency* holds:

(viii) if $\Phi, \Psi, X \in \mathbf{D}_0$, $\Phi \leq \Psi$, and $X \subseteq \Psi$, then $\Phi \cap X \leq X$.

By the definition every generative class $(\mathbf{D}_0; \leq)$ is generated by a set \mathbf{D}'_0 of diagrams in \mathbf{D}_0 such that each $\Phi(A) \in \mathbf{D}_0$ has a copy $\Phi(A') \in \mathbf{D}'_0$.

Let \mathbf{D}'_0 be a class (in particular, a set) of diagrams $\Phi(A)$ over finite sets A , in a language Σ , such that a set of some copies for all elements in \mathbf{D}'_0 is consistent and for each $\Phi(A) \in \mathbf{D}'_0$, $\varphi(\bar{a}) \in \Phi(A)$ or $\neg\varphi(\bar{a}) \in \Phi(A)$ for any quantifier-free formula $\varphi(\bar{x})$ and any tuple $\bar{a} \in A$.

Definition. We say that \mathbf{D}'_0 is *pre-generic*, or *pre-generative*, if it is equipped with a partial order \leq'_0 satisfying conditions (ii), (iii), (vii) for $(\mathbf{D}'_0; \leq'_0)$ as well as the *property of invariance* for \leq'_0 under bijective substitutions as follows:

if $\Phi(A) \leq'_0 \Psi(B)$, $A' \subseteq B'$ and $[\Phi(A)]_{A'}^A, [\Psi(B)]_{B'}^B \in \mathbf{D}'_0$ then $[\Phi(A)]_{A'}^A \leq'_0 [\Psi(B)]_{B'}^B$.

In this case, we also say that $(\mathbf{D}'_0; \leq'_0)$ is *pre-generic*, or *pre-generative*.

By the definition every generative class is pre-generative but not vice versa.

Having the axiom (viii) for a pre-generative class $(\mathbf{D}'_0; \leq_0)$ we also say that $(\mathbf{D}'_0; \leq_0)$ is *self-sufficient*.

Definition. A (pre-)generative class $(\mathbf{D}'_0; \leq_0)$ is *regular* if for any copies $\Phi_1(A_1), \dots, \Phi_n(A_n)$ of elements in \mathbf{D}'_0 with consistent $\Phi_1(A_1) \cup \dots \cup \Phi_n(A_n)$, we have $\Phi_1(A_1) \cap \dots \cap \Phi_n(A_n) = \Phi_i(A_i)|_{(A_1 \cap \dots \cap A_n)}$, $i = 1, \dots, n$.

A (pre-)generative class $(\mathbf{D}'_0; \leq_0)$ is *non-refinable* if for any $\Phi(A) \in \mathbf{D}'_0$ and B with $B \subset A$, there is $\Psi(B) \in \mathbf{D}'_0$ such that $\Psi(B) \subset \Phi(A)$, and $\leq_0 = \subseteq$.

By the axiom of d -uniqueness, every non-refinable (pre-)generative class is regular and self-sufficient. Note that each consistent set of complete diagrams, in a given language, is regular and its closure under complete subdiagrams is non-refinable.

Clearly, if $(\mathbf{D}_0; \leq)$ is a generative class then for any $\mathbf{D}'_0 \subseteq \mathbf{D}_0$ the restriction $(\mathbf{D}_0; \leq)|_{\mathbf{D}'_0}$ is pre-generative. At the same time the following theorem holds:

Theorem 3. *Any regular pre-generative class $(\mathbf{D}'_0; \leq_0)$ can be extended to a non-refinable generative class $(\mathbf{D}_0; \leq)$.*

Proof. Since a set U of some copies of all elements in \mathbf{D}'_0 is consistent, there is a homogeneous model \mathcal{M} of $\bigcup U$. We define $(\mathbf{D}_0; \leq)$ in the following way. Denote by $(\mathbf{D}'_0; \leq')$ the generative class of all complete diagrams over finite sets for models of $\text{Th}(\mathcal{M}|_\Sigma)$. Now we restrict each diagram $\Phi'(A) \in \mathbf{D}'_0$ to the diagram $\Phi(A)$ such that if $\Phi'(A)$ contains a copy $\Psi(B)$ of some $\Psi(B') \in \mathbf{D}'_0$ then $\Phi(A)|_B = \Psi(B)$. The regularity of $(\mathbf{D}'_0; \leq_0)$ guarantees that this restriction is correct and we get a non-refinable generative class $(\mathbf{D}_0; \leq)$ containing $(\mathbf{D}'_0; \leq_0)$, where $\Phi(A) \leq \Psi(B)$ if and only if $\Phi'(A) \leq' \Psi'(B)$. \square

Note that the generative class $(\mathbf{D}_0; \leq)$ in the proof of Theorem 3 depends only on choice of the set U and its model \mathcal{M} . Notice also that if the pre-generative class is not regular then the d -uniqueness property fails in the construction.

At the same time, in the general case, having a pre-generative set \mathbf{D}'_0 with a partial order \leq_0 we can extend $(\mathbf{D}'_0; \leq_0)$ to a generative class $(\mathbf{D}_0; \leq)$ (guaranteeing Axioms (iv), (v), (vi)) if an appropriate diagram $\Phi_0(\emptyset)$ can be added to \mathbf{D}'_0 preserving the pre-generativity, as well as appropriate diagrams $\Theta(D)$ can be added preserving the pre-generativity and step-by-step providing both the d -amalgamation property and the local realizability property.

Minimalism and the Generalization Problem

Cezary Cieśliński

Institute of Philosophy, the University of Warsaw, Poland
E-mail: c.cieslinski@uw.edu.pl

1 Introduction.

According to Horwich's minimalism (see [4]), all the facts about truth can be explained on the basis of the so called 'minimal theory' (MT). The axioms of this theory have the form:

$$(T) \langle p \rangle \text{ is true iff } p$$

where ' $\langle p \rangle$ ' signifies 'the proposition that p '. Horwich claims that the minimal theory fully characterizes the content of the notion of truth. Our understanding of this notion consists in our disposition to accept all (non-paradoxical) instances of (T). In effect, truth becomes unproblematic - it doesn't have any deep nature for the philosophers to uncover.

One of the main concerns for the adherent of Horwichian minimalism is the generalization problem. How can the minimalist account for generalities involving the notion of truth? Consider for example the following generalizations:

- (1) Every proposition of the form ' $\varphi \rightarrow \varphi$ ' is true;
- (2) For every φ , the negation of φ is true iff φ is not true;
- (3) For every φ , φ is true of some object iff it's true that there is an object x such that $\varphi(x)$.

It seems that Horwich's minimal theory, taking as axioms only instantiations of (T), is too weak to prove generalizations of this sort (cf. [2]). In this situation, Horwich owes us an answer to the question: why - if at all - are we entitled to accept them? If the minimal theory doesn't prove generalizations like (1)-(3), how does it help us to arrive at them? This, in a nutshell, is the generalization problem.

In recent years Horwich has made two attempts to deal with this challenge. They will be described below.

First attempt. In [4], Horwich tried to strengthen his minimal theory in such a way that it proves by itself the desired generalizations. The strengthening in question doesn't consist in adding to MT any extra premises, concerning some other subject matter. It consists rather in modifying the proof techniques available to us in MT itself. In Horwich's words:

It is plausible [...] that there is a truth-preserving rule of inference that can take us from set of premises attributing to each proposition some property F, to the conclusion that all propositions have F. ([4], p. 137)

It seems that Horwich is proposing the well known ω -rule as a part of the proof apparatus of MT: if for each proposition φ , $F(\varphi)$ can be derived in MT, then, working in MT, we are entitled to conclude that $\forall\varphi F(\varphi)$. Admittedly, such a move produces a very strong theory and the desired generalizations become theorems of MT.

Second attempt. Unlike in the former case, the current proposal is to leave the proof machinery of MT unchanged (it remains thoroughly classical), but to use it together with a certain additional premise. Horwich stresses (quite correctly) that, apart from MT, the minimalist is permitted to use in his explanations additional ‘truth-free’ assumptions: we can explain for example why we accept ‘<Elephants have trunks> is true’ in MT enlarged with a truth-free assumption ‘Elephants have trunks’. Our understanding of the notion of truth still remains important, but not in isolation - our knowledge of truth-free facts also becomes crucial. In order to deal with the generalization problem, Horwich proposes the following (truth-free) assumption, to be used in explaining our acceptance of truth-involving generalizations:

Whenever someone is disposed to accept, for any proposition of structural type F, that it is G (and to do so for uniform reasons) then he will be disposed to accept that every F-proposition is G. ([5], p. 45)

With this assumption at hand, Horwich promises to explain why we are inclined to accept generalizations of the (1)-(3) type.

As it happens, Horwich’s first solution was quite convincingly rebutted in the literature (see in particular [6]). However, the second strategy seems to be far more promising. In the sections to follow we will describe this strategy in more detail; a certain formal theory will also be proposed as a framework for a Horwichian solution of the generalization problem.

2 Horwichian explanations.

Our aim in this section is quite modest: it’s nothing more than the description of the peculiar properties of Horwichian explanations. To enhance clarity, the characteristic traits of these explanations will be presented here in a very restricted, arithmetical framework. Our starting point is a theory TB^- - a simple, disquotational theory of arithmetical truth. In what follows L_{PA} is the language of first order arithmetic and L_T is the result of extending L_{PA} with a one place predicate ‘ T ’.

Definition 1. $TB^- = PA \cup \{T(\varphi) \equiv \varphi : \varphi \in L_{PA}\}$.

It is a well known fact that TB^- is both arithmetically and truth-theoretically weak; e.g. $TB^- \not\vdash \forall \varphi \in L_{PA} T(\varphi \rightarrow \varphi)$.¹ Given that a disquotational theory like TB^- is our preferred theory of truth, a natural question arises of why we are inclined to accept such general statements. A Horwich-style explanation will be presented below. The explanation is carried out in a metatheory Th about which we assume that:

- (a) The language of Th contains expressions “we realize that ...” and “we are disposed to accept ...”, predicated of sentences.
- (b) Th contains PA.
- (c) Th contains the information that TB is our theory - that we are disposed to accept sentences which we realize to be theorems of TB.
- (d) Th contains the necessitation rule: given φ as a theorem, we are allowed to infer “we realize that φ ”,

¹On the weakness of disquotational truth theories, see [3], especially Proposition 1 on p. 11.

- (e) *Th* contains an axiom to the effect that: if we realize that (for every x , $\varphi(x)$), then for every x , we realize that $\varphi(x)$.
- (f) *Th* contains Horwich's rule: given that we proved ' $\forall x$, we are disposed to accept $\varphi(x)$ ', we are allowed to infer: 'we are disposed to accept $\forall x\varphi(x)$ '.

The explanation proceeds as follows.

Explanation 2.

- (1) For every $\varphi \in L_T$, if we realize that $TB \vdash \varphi$, then we are disposed to accept φ . (By (c))
- (2) For every $\varphi \in L_{PA}$, $TB \vdash T(\varphi \rightarrow \varphi)$. (By (b), since (2) is provable already in PA)
- (3) We realize that (2). (Necessitation, applied to (2))
- (4) For every $\varphi \in L_{PA}$, we realize that $TB \vdash T(\varphi \rightarrow \varphi)$. (From (3) by (e))
- (5) For every $\varphi \in L_{PA}$, we are disposed to accept: $T(\varphi \rightarrow \varphi)$. (From (1) and (4), logic)
- (6) We are disposed to accept: for every $\varphi \in L_{PA}$, $T(\varphi \rightarrow \varphi)$. (By Horwich's rule)

Here are the two traits of the above explanation which I consider worth emphasizing.

Trait 1. The explanation of why we are disposed to accept a given statement could proceed by deriving this statement in a theory which we accept. However, this is not the case here. The general statement in question, i.e. ' $\forall \varphi \in L_{PA} T(\varphi \rightarrow \varphi)$ ', is not derived here at all, neither in TB (which would be impossible, anyway), nor even in TB supplemented with some additional premises. What is derived instead is a statement about our disposition to accept the general sentence under discussion.

Trait 2. The explanation is psychological. A psychological fact (our disposition to accept a given sentence) is explained here in terms of our other dispositions and abilities to realize something.

As for Trait 1, I do not consider it problematic in itself. However, together with Trait 2, it generates the following problem.

Problem. In Explanation 2 the normative element is completely lost and the following additional question arises: is someone who accepts TB (or Horwich's MT, for that matter) in any sense *committed* to accept additional generalizations, unprovable in TB? Assume that we do satisfy the description from Explanation 2. In effect, accepting TB, we are also inclined to accept the general statement ' $\forall \varphi \in L_{PA} T(\varphi \rightarrow \varphi)$ '. But is there a reason why we *should* accept it? (A side remark: explanations which derive the sentence in question from the axioms of some theory accepted by us, would be free of this difficulty!)

In view of this, we are going to propose an alternative approach, which preserves Trait 1, but gets rid of psychological concepts altogether.

3 Believability: an epistemic approach.

We start with the following definition:

Definition 3. Let K be an axiomatisable extension of PA in the language L_K (which is possibly richer than L_{PA}). Denote as $L_{K,B}$ the extension of L_K with a new one place predicate ‘ B ’. Let KB be a theory K formulated in the language $L_{K,B}$.² We denote as $Bel(K)$ the theory in the language $L_{K,B}$ which extends KB with the following axioms:

$$(A_1) \quad \forall \psi \in L_{K,B} [KB \vdash \psi \rightarrow B(\psi)]^3$$

$$(A_2) \quad \forall \varphi, \psi \in L_{K,B} [(B(\varphi) \wedge B(\varphi \rightarrow \psi)) \rightarrow B(\psi)]$$

$$(A_3) \quad \forall \psi \in L_{K,B} \neg B(\psi \wedge \neg \psi).$$

In addition, apart from the usual rules of inference of first order logic, in $Bel(K)$ we are permitted to use necessitation and Horwich’s rule:

$$\text{NEC} \quad \frac{\vdash \psi}{\vdash B(\psi)} \quad \frac{\vdash \forall x B\psi(x)}{\vdash B(\forall x \psi(x))} \quad \text{HR}$$

We will view $Bel(K)$ as a believability theory built over K . The intended reading of ‘ $B(x)$ ’ is ‘ x is believable’; alternatively, we can read ‘ $B(x)$ ’ as ‘there is a good reason to accept x ’. Moreover, a proof of $B(\psi)$ in $Bel(K)$ will be seen by us as showing that ψ should be rationally accepted, given our acceptance of K .

When analysing formal properties of $Bel(K)$, we will be interested in particular in the following question. Assume we start with K as a base theory of our choice. If K is trustworthy, just how trustworthy are the statements which are, provably in $Bel(K)$, within the scope of B ? (Observe that because of the NEC rule, all theorems of $Bel(K)$ are themselves, provably in $Bel(K)$, within the scope of B .) In other words, if $Bel(K)$ declares a statement as believable, will something go awry if we accept it? In order to facilitate the discussion, we adopt the following definition:

Definition 4. $Int_{Bel(K)} = \{\psi \in L_{K,B} : Bel(K) \vdash B(\psi)\}$

The question is now whether the elements of $Int_{Bel(K)}$ form a nice theory. One particular interpretation of the phrase ‘a nice theory’ will be considered here: nice theories are interpretable in the standard model of arithmetic. (It follows in particular that nice theories are ω -consistent.) We are going to claim that $Int_{Bel(K)}$ is indeed nice in this sense. This is the content of the theorem formulated below.

Theorem 5. *If N (the standard model of arithmetic) is expandable to a model N^* of a theory K , then N^* is expandable to a model of $Int_{Bel(K)}$.*

In the proof, the following two notions will be of crucial importance:

Definition 6.

- $S_0 = KB \cup \text{Axioms } (A_1)-(A_3) \text{ of } Bel(K)$,
- $S_{n+1} = S_n \cup \{B(\psi) : S_n \vdash \psi\} \cup \{\forall x \psi(x) : S_n \vdash \forall x B\psi(x)\}$,

²That is, the only difference between K and KB is that KB contains logical axioms also for the formulas with the new predicate.

³Since this axiom contains quantification over all sentences of the extended language $L_{K,B}$, it follows immediately that all the truths of logic in the extended language are within the scope of B - or, as we will say, all the truths of logic in $L_{K,B}$ are believable. In effect (A_1) becomes quite a powerful tool, permitting us to derive e.g. that all instances of induction in $L_{K,B}$ are within the scope of B (cf. [1]).

- $S_\omega = \bigcup_{n \in \mathbb{N}} S_n$.

Definition 7. For a model N^* of K , we define:

- $B_0 = KB$,
- $B_{n+1} = \{\psi : \forall Z \supseteq B_n [\text{if } (N^*, Z) \models (A_2) \wedge (A_3), \text{ then } \rightarrow (N^*, Z) \models \psi]\}$,
- $B_\omega = \bigcup_{n \in \mathbb{N}} B_n$.

The proof of Theorem 5 consists in showing that: (1) $\text{Int}_{\text{Bel}(K)} \subseteq S_\omega$; (2) $(N^*, B_\omega) \models S_\omega$. Then it follows easily that a certain expansion of N^* , namely, (N^*, B_ω) , is a model of $\text{Int}_{\text{Bel}(K)}$.

Having checked that $\text{Int}_{\text{Bel}(K)}$ is a nice theory, we observe that even without the axiom (A_3) , the theory $\text{Bel}(TB^-)$, obtained by taking TB^- as the basic (initially accepted) theory, is already strong enough to prove the believability of all the standard, compositional truth axioms.⁴ Let ‘ $B(CT)$ ’ be a shorthand of ‘all the truth theoretic compositional axioms are believable’. Then we claim that:

Theorem 8. $\text{Bel}(TB^-) - (A_3) \vdash B(CT)$.

In this way a Horwich-style solution to the generalization problem will be vindicated: it turns out indeed that if we find TB^- believable, then we should also find believable various truth theoretic generalizations, independent of TB^- .

In the final part of the proposed talk I will discuss the perspectives of applying a similar argumentation also in the case of some untyped axiomatic theories of truth.⁵

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⁴The compositional truth axioms state that truth commutes with sentential connectives and quantifiers. For example, a part of the proof of Theorem 8 consists in showing that:

$$\text{Bel}(TB^-) - (A_3) \vdash B(\forall\varphi\forall a(\varphi \in \text{Sent}_{L_{PA}} \wedge a \in \text{Var} \rightarrow T(\exists a\varphi) \equiv \exists xT(\varphi(x)))).$$

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Philo v Diodorus Revisited: A New Conditional in an Old Setting, or an Old Conditional in a New Setting?

Richard E. Grandy

Rice University, Houston, TX USA

E-mail: rgrandy@rice.edu

Abstract

Arguments are presented for the merits of many-valued classical logic as a means of shedding light on issues about conditionals and vagueness. A conditional with the properties of a strict implication can be defined in such a system alongside the material conditional, as well as another less familiar conditional. This format also provides more flexibility in understanding Sorites arguments.

1 Introduction

Our standard logic is a considerable success. Although it is generally recognized that it needs supplementation to deal with modalities, tenses and some other natural language phenomena beyond the basics, there is little doubt about it being a correct account of the core of our reasoning. But there is a little doubt. One nagging area of doubt is conditionals. We all know that many students are uncomfortable with some of the logical truths, such as $(A \supset B) \vee (B \supset C)$ and $\sim A \supset (A \supset C)$. At a more professional level of discomfort we observe that there are dozens of books on conditionals, but none on conjunctions or disjunctions. Or we can note that the disagreement about the proper theory of conditionals is 2300 years old. Diodorus Cronus proposed understanding the natural language Greek conditional as a material conditional; his student Philo of Megara objected because of theorems like those just cited and argued for what is essentially the S5 strict implication as the correct analysis.

Another ancient area of continuing philosophical investigation which so far has mainly achieved proliferation of theories rather than convergence is vagueness and the Sorites paradox. It isn't even clear where to locate that problem in philosophical space—is it an issue of logic? Metaphysics? Epistemology? Pragmatics?

In this paper I argue for looking into many-valued classical logics because it provides space for alternative accounts of conditionals and a richer theory of vagueness within classical logic.

2 Many-valued Classical Logic

But you complain, “Classical Logic is two-valued!”, and giving up bivalence would mean changing our logic and wandering into the dark and stormy many-valued night. However, we have to distinguish two meanings of the sentence “Classical Logic is two-valued”, one true and one false.

True: It is a fact, proved in every intermediate logic course that a sentence is derivable in one of the standard sentential logic systems iff it is valid on the usual two-valued truth interpretations.

True: In the most frequently discussed many-valued logics, e.g., Lukasiewicz or intuitionistic logics, some classically valid sentences are not valid on those interpretations.

False: In any kind of interpretation with more than two truth values, the valid sentences differ from the classic ones in two-valued logic.

True: In Boolean-valued logics, which can have 2^n values for any positive integer n (as well as some infinite interpretations), exactly the classically two-valued valid sentences are valid in these interpretations. No familiar tautologies need be surrendered.

For reasons of familiarity and relative concreteness, I present the argument and definitions in terms of set-valued logic. Set-valued logic is one kind of Boolean-valued logic and will serve my purposes here.

We assume the usual array of sentence letters A, B, \dots and the familiar logical operations, including the material conditional \supset . However we also add symbols for new conditionals: \rightarrow and \rightarrow . A model \mathbf{M} of our language utilizes a non-empty set \mathbf{T} and the values assigned are subsets of \mathbf{T} . In other words, a model \mathbf{M} assigns a subset of \mathbf{T} to each atomic letter. We extend the model to non-atomic sentences with the familiar connectives by familiar clauses; a model assigns to a conjunction the intersection of the assignments to the conjuncts, the union to the disjunction and so on. $\mathbf{M}(\Phi \supset \Theta) = (\mathbf{T} - \mathbf{M}[\Phi]) \cup \mathbf{M}[\Theta]$.

In addition, for \rightarrow we let the model \mathbf{M} assign to $(\Phi \rightarrow \Theta)$ the entire set \mathbf{T} iff $\mathbf{M}[\Phi] \subseteq \mathbf{M}[\Theta]$, and the empty set otherwise.

This conditional has the properties of S5 strict implication. It entails the corresponding material conditional, but is not entailed by it. Many familiar and plausible conditional principles hold, e.g., modus ponens and transitivity, and many dubious ones fail. None of $(A \rightarrow B) \vee (B \rightarrow A)$, $(A \rightarrow B) \vee (B \rightarrow C)$ nor $(\sim A \rightarrow (A \rightarrow C))$ are valid. Unfortunately, $(C \rightarrow (A \rightarrow C))$ and $A \rightarrow (B \rightarrow (A \& B))$ are also not valid and those are principles most people accept. So it appears that we need a conditional intermediate between the material and the strict implication.

Many-valued semantics gives us room for such an intermediate. We let model \mathbf{M} assign to $(\Phi \rightarrow \Theta)$ the entire set \mathbf{T} iff $\mathbf{M}[\Phi] \subseteq \mathbf{M}[\Theta]$, and $\mathbf{M}[\Theta]$, otherwise. This conditional is intermediate in that $(\Phi \rightarrow \Theta)$ entails $(\Phi \supset \Theta)$, which in turn entails $(\Phi \supset \Theta)$.

Again none of $(A \rightarrow B) \vee (B \rightarrow A)$, $(A \rightarrow B) \vee (B \rightarrow C)$ nor $\sim A \rightarrow (A \rightarrow C)$ are valid. But happily $(C \rightarrow (A \rightarrow C))$ is. Unfortunately, we still can't quite accommodate all the desirable intuitions since $A \rightarrow [B \rightarrow (A \& B)]$ is not valid. Notice that if we restrict ourselves to a two-valued interpretation, the three conditional connectives coincide in that context. Thus at this point we have not found the ideal version of a conditional intermediate between the material and the strict, but I hope you can see why I believe this area is worth further exploration.

But first some philosophical objections:

One natural objection is that we have no clear idea of either how many truth values we should be using (except that it should be 2^n for some n), or what they mean. Let's divide the objection into the two natural parts, one about how many and the other about what they are. Not knowing with philosophical precision what the truth values are would be a serious objection if we had a clear and agreed upon understanding of the two classical truth values. But we don't. The most common view, Correspondence, has well known problems once we move beyond the atomic sentences, and fragments into a host of competing accounts of what Correspondence means. And a significant minority of philosophers hold non-correspondence views, most commonly Deflationary in recent decades. The main point of the Deflationary view is readily adapted to a multiplicity of truth values as long as you let $\text{True}(\Phi)$ have the same value as Φ .

As to the number of truth values, I propose an exact but context dependent answer: For any text or argument analyzed in sentential logic with n atomic sentences, 2^n truth values suffice. For analysis in quantification theory, the number is a more complex function of the number of

predicates, relations and their degree, and the size of the domain. In the early 20th century, as formal logic developed logicians moved from the conception that logic has one fixed domain to recognize that part of specifying a model includes specifying a domain. I suggest we need to make a similar move and recognize that part of specifying a model is specifying how many truth-values the model will deploy.

3 Sorites and Vagueness

Lukasiewicz' many-valued logics are one alternative to classical logic that may seem to offer a satisfactory treatment of Sorites issues, but in fact they do not. Consider the following Sorites:

Premise 1: **One stone is not a heap.**

Premise 2: **If one stone is not a heap, then two stones are not a heap.**

Premise 3: **If two stones are not a heap, then three stones are not a heap.**

...

Premise $n + 1$: **If n stones are not a heap, then $n + 1$ stones are not a heap.**

...

Premise 100: **If 99 stones are not a heap, then one hundred stones are not a heap.**

Conclusion: **One hundred stones are not a heap.**

In Lukasiewicz logics it is reasonable to give some intermediate truth value, probably .99, to each conditional premise. That gives a fairly satisfactory starting point because each of those seems pretty nearly true but not entirely. The problem is that when we take the conjunction of the 99 conditional premises with truth value .99, we have a conjunction which also has truth value .99 but we have a conclusion with truth value zero.

On the issues we have been discussing with standard truths involving conditionals, Lukasiewicz logics don't validate $(A \supset B) \vee (B \supset C)$ but do validate $(A \supset B) \vee (B \supset A)$. The validity of the latter follows from the fact that the Lukasiewicz values are linearly ordered. Reflecting on various forms of vagueness, I would argue that while some vague terms involve a single dimension, perhaps height, many are multidimensional and it would a mistake to build into our account of vagueness a strict ordering of degrees of vagueness.

In contrast, set-valued logics offer a solution to at least this aspect of Sorites because the truth value of a conjunction, defined as the intersection of the values of the conjuncts, can be less than the truth value of either component. Thus the conjunction of the 99 Sorites premises mentioned above can be much falser than any one of the single components. Looking at the dual situation, one could argue that some disjunctions are more true than either disjunct. Certainly natural language speakers are often more confident that something is blue-or-green than they are that it is blue or that it is green. Note that the Sorites conditionals only have varying degrees of truth if we translate them as material conditionals or \rightarrow . Taken in terms of \neg they are each maximally or minimally true.

4 What is Logic about Anyway?

According to Boghossian and Peacock (2000), we know *a priori* that $(A \supset B) \vee (B \supset C)$. But what does that mean? Is it that we know $(A \supset B) \vee (B \supset C)$ always has value 1 given the way \supset is defined in our texts and elsewhere? That seems like knowing 7 is odd and prime, a mathematical fact, not a matter of logic. We also know just as *a priori* that it does not always have value 1 in Lukasiewicz logics (Grandy, 1993).

So perhaps what we know is the additional fact that classical mathematics correctly portrays the structure of reality. Or that it is a useful model of the structure of reality? Or of the rules of English? Or of the practices of English speakers/reasoners? Or of the cognitive processes of English speakers? Or the practices of idealized English speakers? Or the cognitive processes of the ideal English speaker? These are very deep issues and I don't have a short answer, but my point is to argue for another candidate for capturing or modeling whatever is being captured or modeled.

I also want to recognize that speakers may not be using a single interpretation of "if then". Just as "or" may sometimes be inclusive and sometimes exclusive, perhaps "if then" is naturally sometimes interpreted as \supset , sometimes as $\supset\text{-}$ and at others as \rightarrow . This would explain some of the persistence of the dispute. And there is at least one imaging study of brain functioning that lends some support to the suggestion by showing that different forms of conditional problems are processed in different brain areas (Osherson et al., 1998).

Evaluating formal models of natural language terms is very complex. There are at least two kinds of complication: Pragmatics and Human Error.

Pragmatic contextual Gricean conversational principles interfere with evaluation. Speakers may balk at the claim that "If A then C" is true when A is known to be false, but that may be because the conditional is less informative than " \sim A and C" and is thus conversationally *inappropriate*, not because they judge it false (Grice, 1989).

And there is ample experimental evidence that humans make mistakes in judgments about conditionals. Consider the "Four Card Experiment". Given a special set of cards which have a letter on one side and a positive integer on the other, subjects are shown one side of each of four cards and asked whether they need to turn over that card in order to check whether the cards follow the rule: "If there is a vowel on one side, there is an odd number on the other". The cards show:

A M 7 4.

Subjects reliably say they need to turn over the A card, and don't need to turn over the M and 7 cards, but many subjects do not see that they need to turn over the 4 card. I have replicated this myself on an intermediate logic examination where a third of the class made that error. And it is an error. Whether the rule is interpreted with the any of the three conditionals discussed above or the intuitionist or Lukasiewicz conditional, the subjects are wrong.

5 Conclusion

I suggest we have four hypotheses about the situation, none obvious and none excluded (and not exhaustive).

Univocal Optimism– There is a conditional definable in many-valued classical logic that captures a univocal sense used by natural language speakers as their understanding of if-then.

Ambiguity Optimism– People sometimes use one conditional and sometimes another, but we can sort these out into consistent patterns captured within our framework.

Pessimism about Cognizers– People have intuitions that partly coincide with various conditionals, but there is no coherent story to get right.

Pessimism about Valuations– The best theory is that some conditionals don't have truth values. (Edgington, Grandy & Osherson)

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Dialetheism and Dual-Valuation Logics

Ben Martin

University College London, London, U.K.

E-mail: Benjamin.martin@ucl.ac.uk

Abstract

Since Priest (2006b) proposed dialetheism, the thesis that some contradictions are true, as a solution to self-referential paradoxes, considerable research has been conducted not only on the success of these dialethic solutions, but on the ability of dialethic semantics to express certain meaningful claims and account for the success of certain dialethically invalid inferences. This paper discusses two such concerns over current dialethic semantics and offers a new family of zero-order logics, dual-valuation logics, of which one member **gD-V** provides dialethic semantics that resists both concerns.

1 Paraconsistency and Dialethic Semantics

A logic L is paraconsistent if and only if $\{A, \sim A\} \not\vdash_L B$. Call a logic that not only invalidates this unconjoined form of explosion, but the conjoined form $\{A \wedge \sim A\} \vdash B$, ‘strongly paraconsistent’. Some paraconsistent logics are strongly paraconsistent, and others are not. For example, while both Jaśkowski’s (1969) discursive logic and Jennings and Schotch’s (1984) preservationist logic are paraconsistent, neither are strongly paraconsistent. Now, call a logic in which contradictions, formally conceived as formulae of the form $A \wedge \sim A$, can be assigned the truth-value true ‘dialethic’. Unsurprisingly, any dialethic logic worth its salt is also paraconsistent and strongly paraconsistent, as any dialethic logic which isn’t will be trivial.

As the dialetheist proposes the truth of some contradictions, while rejecting trivialism, she requires a strongly paraconsistent dialethic logic. In addition, for philosophical reasons, dialetheists prefer both logics that respect the normal semantics for conjunction and negation (Priest & Routley, 1989), that is $v(A \wedge B) = \min\{v(A), v(B)\}$ and $v(\sim A) = 1 - v(A)$, and logics that conceive of propositional valuations as relations between propositional parameters and the set of truth-values, not functions (Priest, 1996). This ensures that dialetheists have been wary of using da Costa’s **C ω** logics, as they provide non-normal semantics for negation. The classic example of a non-trivial dialethic logic that fulfils these conditions is Priest’s **LP**, as later conceived with valuation relations in Priest (1995).

Now, unfortunately for the dialetheist, although there are logics, such as **LP**, that possess the properties she requires, any logic which fulfils all of these conditions is susceptible to two troublesome criticisms. Firstly, the logics cannot express the concepts of ‘true only’ or ‘false only’ and, secondly, the logics cannot account for the quasi-validity of certain classically valid inferences.

2 The Inexpressibility of ‘True only’

Any dialethic logic respecting the normal semantics for conjunction and negation must allow for propositions to be assigned both truth-values. This ensures that the dialetheist cannot express disagreement with anyone over the truth or falsity of a proposition by stating that the proposition is false or true, respectively, as asserting the truth or falsity of a proposition doesn’t preclude its being, respectively, false or true also. Consequently, to express disagreement over the truth of a

proposition the dialetheist must rely upon the speech acts of assertion and denial (Priest, 2006a: Ch. 6), with denial interpreted as a *sui generis* speech act that rules out the speaker's assertion, and thus endorsement, of the proposition.

However, as Shapiro (2004) has recognized, there are plenty of cases in which the speech-acts of assertion and denial can't adequately replace the semantic concepts 'true only' or 'false only'. For example, if one wants to suppose that a proposition p is false only and derive the consequences of this, this cannot be achieved with the denial of p , for force operators cannot meaningfully be embedded into truth-functional contexts such as conditionals. Consequently, a dialethic semantics must be capable of expressing that a proposition p is true only or false only if it's to be able to accommodate certain meaningful conditionals.

Additionally, given that a common motivation for a dialethic semantics is the inability of classical semantics to account for certain meaningful linguistic phenomena, such as the Liar sentence, it would seem a weakness of the dialethic research programme if meaningful expressions such as 'Proposition p is true only' and 'Proposition p is false only' could not be expressed within a dialethic semantics. Consequently, for the dialethic research programme to adhere to its own standards, it should have the resources to express the concepts of 'true only' and 'false only'. At present, however, none of the dialethic logics we are interested in here can express these concepts. **LP**, for example, only has the ability to communicate that a propositional variable p has a valuation relation to a truth-value t , $p\varepsilon t$, which doesn't preclude it also having a valuation relation to another truth-value.

3 Quasi-Valid Inferences

Dialethic logics that respect the normal semantics for conjunction and negation invalidate certain classically valid inferences. Consider, for example, the case of modus ponens $\{A, A \supset B\} \vdash B$ in **LP**. A countermodel can easily be produced by assigning A both truth-values ($A\varepsilon 1$ and $A\varepsilon 0$) and falsity to B ($B\varepsilon 0$). Dialetheists recognize, however, that such inferences are valid in consistent situations, calling such inferences 'quasi-valid'. Classical logic is thus seen as a limiting case of dialethic semantics when the propositions in question behave consistently, just as classical logic is a limiting case of intuitionist logic when the context in question is decidable.

Yet if dialethic semantics are conceived as possessing more expressive power than its classical counterpart, with dialethic semantics subsuming classical semantics as an inadequate precursor, then dialethic semantics must be able to express and accommodate these 'quasi-valid' inferences. In other words, if it's acceptable to use these (very important) quasi-valid inferences in consistent situations, then our semantics better be able to model those situations in which we are permitted to use them. Thus, if a dialethic semantics are to fully supersede classical logic, it needs to be able to accommodate all the successes of classical logic itself, which includes being capable of expressing cases in which it's acceptable to infer according to the quasi-valid inferences.

4 Dual-Valuation Logics

Ideally for the dialetheist's sake there should be a strongly dialethic semantics available that avoids both of these concerns while respecting the normal semantics for conjunction and negation, and modelling zero-order valuations as relations between propositional parameter and truth-values. This paper proposes that there is just such a logic available to the dialetheist, **gD-V**. Before we consider **gD-V**, however, we need to introduce the family of logics of which **gD-V** is a member, the *dual-valuation* logics.

Dual-valuation logics are logics which possess two relations, ε^+ and ε^- , that relate propositional parameters to truth-values, rather than one. The valuation relation ε^+ is the same relation as is found in other logics with relational semantics, such as **LP**. The new relation ε^-

found in dual-valuation logics, however, symbolizes the logic's *anti-valuation* relation, instead of the logic's *valuation* relation. To understand the function of the two relations, it's helpful to consider an analogy with the extension and anti-extension of a predicate. Just as certain objects are in the extension of a predicate P , and other objects are in P 's anti-extension, so truth-values are either in the *valuation* set of a proposition p , or its *anti-valuation* set. While the *valuation* set of p is dictated by the truth-values that p has the relation ε^+ to, the *anti-valuation* set of p is dictated by the truth-values the proposition has the relation ε^- to. As before, valuations are relations from propositional parameters to the set of truth-values $\{1, 0\}$, and anti-valuations are similarly relations from propositional parameters to the set of truth-values $\{1, 0\}$.

There are two restrictions which all members of the dual-valuation family of logics place upon the membership conditions of the valuation and anti-valuation sets for a proposition. For any proposition p and truth-value t :

- 1) Either $p\varepsilon^+t$ or $p\varepsilon^-t$, and
- 2) It's not the case that both $p\varepsilon^+t$ and $p\varepsilon^-t$.

Thus, it's an assumption of all dual-valuation logics that the valuation and anti-valuation sets partition the set of truth-values $\{\text{true}, \text{false}\}$ for every propositional parameter. Consequently, these restrictions placed on the valuation and anti-valuation sets ensure that the metatheory of the logics behave consistently. Given this, it's reasonable to conceptualize the anti-valuation relation as communicating which truth-values are *not*, classically understood, members of the valuation set of a proposition p .

Any other restrictions placed upon the membership conditions of the valuation and anti-valuation sets for a proposition are dependent upon the individual dual-valuation logics. The dual-valuation logic of particular interest to us here is the gluttony but non-gappy **gD-V**. In **gD-V** both truth-values can be members of a propositional parameter's valuation set, however the valuation set for a propositional parameter must be non-empty. Thus, although a propositional parameter p can have the valuation relation to both true and false, p can only have the *anti-valuation* relation to either true or false. Consequently, there are three permissible total valuations for a propositional parameter in **gD-V**: 1) $p\varepsilon^+1$ and $p\varepsilon^-0$; 2) $p\varepsilon^+0$ and $p\varepsilon^-1$; and 3) $p\varepsilon^+1$, $p\varepsilon^+0$, and $p\varepsilon^-\emptyset$.

By providing the dual truth-conditions of conjunction and negation for the valuation and anti-valuation relations, we can show that **gD-V** is a strong paraconsistent dialethic logic:

$$\begin{aligned} (A \wedge B)\varepsilon^+1 &\text{ iff } A\varepsilon^+1 \text{ and } B\varepsilon^+1 \\ (A \wedge B)\varepsilon^+0 &\text{ iff } A\varepsilon^+0 \text{ or } B\varepsilon^+0 \\ (A \wedge B)\varepsilon^-1 &\text{ iff } A\varepsilon^-1 \text{ or } B\varepsilon^-1 \\ (A \wedge B)\varepsilon^-0 &\text{ iff } A\varepsilon^-0 \text{ and } B\varepsilon^-0 \end{aligned}$$

$$\begin{aligned} (\sim A)\varepsilon^+1 &\text{ iff } A\varepsilon^+0 \\ (\sim A)\varepsilon^+0 &\text{ iff } A\varepsilon^+1 \\ (\sim A)\varepsilon^-1 &\text{ iff } A\varepsilon^-0 \\ (\sim A)\varepsilon^-0 &\text{ iff } A\varepsilon^-1 \end{aligned}$$

Let the propositional parameter p have the valuation relation to both truth and falsity, $p\varepsilon^+1$ and $p\varepsilon^+0$. Now we have both $p\varepsilon^+1$ and $\sim p\varepsilon^+1$, and consequently $(p \wedge \sim p)\varepsilon^+1$. Thus, contradictions can be assigned the truth-value true, while retaining the intuitive consequence that contradictions are false under every valuation, $(p \wedge \sim p)\varepsilon^+0$. **gD-V** is a dialethic logic. To show that **gD-V** is also paraconsistent and strongly paraconsistent, consider **gD-V**'s consequence relation,

$$\Sigma \vDash_{\text{gD-V}} B \text{ iff for all } \varepsilon^+ \text{ and } \varepsilon^-, \text{ if } A\varepsilon^+1 \text{ for all } A \in \Sigma, \text{ then } B\varepsilon^+1,$$

and allow p to have the valuation relation to both truth and falsity. As before, this ensures we have both $p\varepsilon^+1$ and $\sim p\varepsilon^+1$, and consequently $(p \wedge \sim p)\varepsilon^+1$. This is enough to invalidate both the

unconjoined and conjoined forms of explosion. Consequently, $\mathbf{gD-V}$ is a strongly paraconsistent dialethic logic.

We can now show that $\mathbf{gD-V}$ fails to suffer from our two concerns over current dialethic semantics. Let us begin by showing that $\mathbf{gD-V}$ can express that a proposition p is ‘true only’ or ‘false only’, with the introduction of consistent truth^{CT} and consistent falsity^{CF} operators (we will only provide the valuation truth-conditions for the operators here, as the anti-valuation conditions are redundant for our purposes):

$$\begin{aligned} (^{CT}A)\varepsilon^+1 &\text{ iff } A\varepsilon^-0, \text{ otherwise } (^{CT}A)\varepsilon^+0 \\ (^{CF}A)\varepsilon^+1 &\text{ iff } A\varepsilon^-1, \text{ otherwise } (^{CF}A)\varepsilon^+0. \end{aligned}$$

Thus, for any proposition p , we can express that p is true only, ^{CT}A , or that p is false only, ^{CF}A . Both the consistent truth and consistent falsity operators can be shown to behave consistently in $\mathbf{gD-V}$. Under no valuation is it the case that for some A , both $(^{CT}A)\varepsilon^+1$ and $(^{CF}A)\varepsilon^+1$, both $(^{CT}A)\varepsilon^+1$ and $\sim(^{CT}A)\varepsilon^+1$, or both $(^{CF}A)\varepsilon^+1$ and $\sim(^{CF}A)\varepsilon^+1$. I leave proof of this to the reader.

Similarly, just as we introduced a consistent truth and consistent falsity operator into $\mathbf{gD-V}$, so we can introduce a consistency operator^C which communicates that the subsequent formula is either consistently true or consistently false,

$$(^CA)\varepsilon^+1 \text{ iff } A\varepsilon^-0 \text{ or } A\varepsilon^-1, \text{ otherwise } (^CA)\varepsilon^+0.$$

Having shown that $\mathbf{gD-V}$ contains consistency operators, we can now give a precise characterization of quasi-valid inferences within $\mathbf{gD-V}$ as those inferences that are invalid within $\mathbf{gD-V}$, but are valid within $\mathbf{gD-V}$ on the assumption that if true, the members of the premise set are *consistently* true: $\Sigma \not\models_{\mathbf{gD-V}} B$ but $^{CT}(\Sigma) \models_{\mathbf{gD-V}} B$. Consequently, the validity of any quasi-valid inference, that is, any *classically* valid inference, can be recaptured within $\mathbf{gD-V}$ by adding the assumption that all the members of the premise set behave consistently. Again, I leave the proof of this to the reader.

The ability of $\mathbf{gD-V}$ to recapture classical validity through consistency assumptions in the logic’s object-language ensures that it is a *Logic of Formal Inconsistency (LFI)*. As an **LFI**, an explosive negation can be introduced into $\mathbf{gD-V}$ so that the logic contains both an unexplosive and explosive negation, as well as an explosive bottom particle (Marcos, 2005). Detailing these, and other, interesting properties of $\mathbf{gD-V}$, however, is beyond the scope of this abstract.

Another interesting result, which it is beyond the scope of this abstract to detail, is that an intuitive modal extension of $\mathbf{gD-V}$, $\mathbf{gD-Vm}$, can block the dialetheist’s commitment to the impossibility of the actual world. In Martin (2015) it’s been shown that a modal extension of any dialethic logic with the normal semantics for conjunction and negation, using standard modal semantics, commits the dialetheist to the impossibility of the actual world. As all contradictions are false in such dialethic logics, although some are also true, $\sim(p \wedge \sim p)$ is a theorem of these dialethic logics. Thus, by necessitation we can derive $\Box \sim(p \wedge \sim p)$, which given the interdefinability of the necessity and possibility operators ensures we can derive $\sim\Diamond(p \wedge \sim p)$. However, interpreted naturally, this formula reads as ‘It’s impossible for a contradiction to be true’. Given that an impossible world is a world w at which propositions it’s impossible to be true are true, any world w at which a contradiction is true is going to be an impossible world according to these modal semantics. Consequently, given that the dialetheist endorses the truth of some contradictions at the actual world, the logic subsequently commits her to the actual world being an impossible world. Thus, by simply assuming necessitation and the interdefinability of the modal operators, it can be shown that a modal extension of the dialethic logics we are concerned with here commit the dialetheist to the impossibility of the actual world. Given the implausibility of such a commitment, it might seem theoretically preferable to build a dialethic logic the modal extension of which doesn’t commit the dialetheist to the impossibility of the actual world. After all, if one is asserting a controversial thesis, it is a theoretical virtue to not assert an additional controversial thesis if it isn’t necessary. Consequently, $\mathbf{gD-V}$, and its intuitive modal extension

gD-Vm, can be shown to provide a semantics that resist three concerns over current dialethic semantics.

5 Conclusion

In this abstract we have discussed two weaknesses of current dialethic semantics that use valuation relations and respect the normal semantics for conjunction and negation: their inability to express that a proposition is only true or only false, and their inability to accommodate quasi-valid inferences. A new propositional logic, **gD-V**, which is a member of the dual-valuation family of logics has been proposed which fails to suffer from either of these weaknesses. Additionally, it has been suggested that an intuitive modal extension of **gD-V** blocks the dialetheist's commitment to the impossibility of the actual world, a consequence of current dialethic semantics demonstrated elsewhere.

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Supervaluation and Revision

Edoardo Rivello

University of Torino, Torino, Italy
E-mail: rivello.edoardo@gmail.com

1 Semantic theories of truth

Any theory of self-referential truth has to deal with the so-called *liar paradox*, namely with the coexistence of two intuitively convincing claims about truth which actually contradict each other: the first one is that our use of the word ‘true’ seems to imply the validity of all sentences (called *Tarski-biconditionals*) of the form

‘ ϕ ’ is true if and only if ϕ ,

where ϕ represents any sentence of the language; the second one is that there are sentences (like the *liar sentence*, denoted by λ), which are paradoxical, in the sense that to assume the validity of the corresponding Tarski-biconditional for λ leads to a contradiction.

Inspired by Tarski’s definition of truth (for languages which do not involve their own truth predicate), *semantic theories of truth*¹ aim to provide an interpretation, in the metalanguage, for a truth predicate added to the object language. Formally, let \mathcal{L} be a first-order language and suppose that all non-logical symbols of \mathcal{L} are interpreted in some model, and let T be a unary predicate added to \mathcal{L} . Tarski’s definition of “truth-in-a-model” can be viewed as the standard way of determining which sentences of the expanded language $\mathcal{L}_{\mathsf{T}} = \mathcal{L} \cup \{\mathsf{T}\}$ are “true” *provided* an interpretation for T . We can formalise this as an operator Γ (the *Tarskian operator*) which takes as argument a subset X of sentences of \mathcal{L}_{T} (intended as a possible interpretation for T) and returns another set $\Gamma(X)$ of sentences of \mathcal{L}_{T} , namely those which become true according to Tarski’s rules under the given interpretation for T . The ideal solution (one making all Tarski’s biconditionals hold) should be an interpretation X of T such that $\Gamma(X) = X$. Unfortunately, the liar paradox tells us that such a solution cannot exist. Consequently, semantic theories of truth provide an interpretation for the truth predicate aiming to preserve some of the properties of Tarski’s definition though avoiding the liar paradox.

Many semantic theories of truth fall under one of two major paradigms: *fixed-point semantics*, sketched by Kripke [6] in 1975, and *revision semantics*, this latter introduced by Herzberger and Gupta, independently, in 1982².

The fixed-point approach consists in replacing the Tarskian operator Γ by a *monotone operator* Δ which takes as argument one *partial interpretation* p for the truth predicate T and returns a *partial evaluation* of the sentences of \mathcal{L}_{T} . Both partial interpretations and partial evaluations can be formalised as partial characteristic functions, namely functions which, to every member of some set of sentences of \mathcal{L}_{T} , assign one of the two truth values **t** (for ‘true’), or **f** (for ‘false’). Any monotone operator Δ has (many) *fixed-points*, namely partial interpretations p such that $\Delta(p) = p$, which can be used to study the behaviour of the truth predicate for different kinds of sentences. In particular, each fixed-point can be viewed as a distinguished partial interpretation of the truth predicate, since the Tarski biconditionals hold for each sentence in the domain of the fixed-point.

¹See [4] for a recent overview.

²The standard reference for revision semantics is [2].

The revision-theoretic approach, on the other hand, is the idea that the Tarski biconditionals provide a “rule of revision” of our naïve insights about truth. Formally, this idea is captured by the Tarskian operator Γ itself, whose application can be iterated as follows: given a sentence ϕ in which \top occurs and given a hypothesis h about the interpretation of \top (namely, about which sentences are true and which one are false), we can apply the Tarskian operator and evaluate the truth value of ϕ under the hypothesis h . The so-calculated truth-values for all sentences of the language give a new hypothesis h' which “revises” the original one by changing the truth value of some sentences. Different starting hypotheses generate different *revision sequences* (ordinal-indexed iterations of the Tarskian operator) which can be used, instead of fixed-points, to study the behaviour of the truth predicate for different kinds of sentences. In particular, each sentence which “survives” all revision processes (namely, which eventually gets the same truth value in all revision sequences, for every starting hypothesis) is declared to be *stable*, and a set of stable sentences can be taken as a distinguished partial interpretation of the truth predicate, alternative to those arising from the fixed-point approach.

Both approaches are characterised by the transfinite iteration of an operator which evaluates the sentences of \mathcal{L}_\top given a hypothesis about which sentences are true and which ones are false. We can summarise the main differences between the two kinds of iterations in the following table:

Fixed-point semantics	Revision semantics
Partial interpretations	Total interpretations
Monotone operator	Tarskian operator
Approximating sequences	Revision sequences
Fixed-points	Sets of stabilities

The fixed-point approach allows many different formal fixed-point semantics, due to the arbitrariness in the choice of the evaluation scheme to apply and of the fixed-points we want to consider. Similarly, the revision-theoretic approach can be instantiated in many different formal revision semantics due to the arbitrariness in the choice of the starting hypotheses and of the hypotheses we want to consider at the limit stages of the revision process.

2 Revision-theoretic supervaluation

Very often, in the literature the fixed-point semantics and the revision-theoretic semantics are contrasted as competitors. My current project takes a different stance, aiming to provide a formal semantic theory of truth which could combine virtues from both fixed-point and revision approaches.

The general idea, roughly speaking, is to use the revision-theoretic notion of “stable truth” in order to extend any partial interpretation of the truth predicate to a partial evaluation of the sentences of \mathcal{L}_T and then to perform a fixed-point construction in Kripke’s style. Formally, this can be done by defining a monotone operator through a variant of the van Fraassen’s *supervaluation* scheme. Let \mathcal{C} be a collection of revision sequences (either of ordinal or limit length). Given a partial interpretation p for the truth predicate \top we pick one set H_p whose members are total interpretations extending p . Then we consider the partial evaluation $\Delta(p)$ given by those sentences which are declared to be *stably true* (or *stably false*) by all revision sequences in \mathcal{C} which start with a hypothesis in H_p . The operator Δ so defined turns out to be monotone, so it makes sense to study the structure of its fixed-points.

The leading motivation is to take seriously Gupta’s notion of revision-theoretic *improvement* and to base a formal notion of revision on it:

When we learn the meaning of ‘true’ what we learn is a rule that enables us to improve on a proposed candidate for the extension of truth [1, p. 37].

In the literature it has been observed many times, even by Gupta himself, that the usual formalisation of revision is not able to capture any informal notion of “improvement”

The term *revision process* suggests that by applying the [Tarskian] operator better and better extensions for the truth predicate [...] are obtained. But neither is it clear what *better* could mean here nor is it plausible to assume that the revision process leads to better models as one may start with a good model and then apply the revision process to this model [3, p. 164].

By taking the notion of stable-truth as constitutive of a supervaluational fixed-point theory, we point to two goals: one is to maintain the idea of “revision” captured by the iteration of the Tarskian operator; and the other is to formalise the idea of “improvement” by the monotone process which gets larger and larger partial interpretations for the truth predicate.

Many variants of the fixed-point semantics sketched above could be considered, simply by combining a particular choice for the set of the starting hypotheses H_p (as in well studied variants of van Fraassen’s supervaluation) with one collection of revision sequences \mathcal{C} , this latter representing the choice of a particular revision-theoretic semantics. For instance, taking H_p to be the set of *all* total interpretations extending p (as in the original supervaluational fixed-point semantics suggested by Kripke) and taking \mathcal{C} to be the collection of *all* ordinal-length revision sequences (as in Belnap-Gupta revision semantics described in [2]) we obtain a monotone operator Δ whose least fixed-point is reached after just one iteration (starting with the empty interpretation) and which coincides with the set of stabilities in Gupta-Belnap sense. However, other choices of H_p and \mathcal{C} can give more interesting semantics, both from a mathematical and from a philosophical point of view.

An especially intriguing choice for \mathcal{C} is the set of all ω -length iterations of the Tarskian operator. Such iterations actually correspond to a strong intuitive notion of revision. A finite or denumerable sequence of repeated applications of the Tarskian operator can be easily understood as the formal counterpart of a (idealised) revision process, namely a mode of reasoning performed by an agent which revises previous hypotheses by means of a rule of revision represented by the Tarskian operator. Unfortunately, it is well-known that finite or countable revision sequences lead to an unsatisfactory theory of truth, namely to a theory which makes counterintuitive predictions about which sentences are to be evaluated as (stably) true, (stably) false or paradoxical.

For this reason, formal revision theories of truth are usually based on *transfinite* (or even *ordinal-length*) iterations of the Tarskian operator. However, this move seems to lose the clear insight of a revision process by introducing arbitrary choices at the limit stages, as argued, among others, by Halbach:

The revision process at successor levels is very well understood and appraisals of the revision theory often focus on the definition of successor levels, while the definition of limit levels is somewhat artificial [...] Concentrating on the finite levels of the revision process is worthwhile: one can thereby avoid all the difficult issues concerning limit levels and just capture the chief attractive feature of revision semantics, which is the revision process via the [Tarskian] operator [3, p. 167, 168].

The hope is that, by combining ω -revision with the supervaluational fixed-point approach, on one side we can preserve the strong intuitive sense of “revision” which is captured by the iteration of the Tarskian operator and, on the other side, we can reach the “correct” predictions about stability made by the transfinite revision sequences.

Some preliminary results, obtained by elaborating on previous works about revision as in [8] and [7], suggest that the present proposal, namely the idea of performing fixed-point semantics by a supervaluational scheme based on ω -revision, could in fact formalise an interesting theory of truth. For the sake of brevity let me call the present proposal *ω -revision supervaluation*. Then we have the following facts:

- There are examples of sentences which requires transfinite stages (in order to be declared revision-theoretically stable) and which are correctly declared non-paradoxical by ω -revision supervaluation³.
- Each grounded sentence (in the sense of Van Fraassen’s supervaluation) is also grounded in the sense of ω -revision supervaluation.
- There are examples of sentences which are grounded in the sense of ω -revision but not in the sense of van Fraassen’s supervaluation⁴.
- Each grounded sentence in the sense of ω -revision supervaluation is nearly stable in all revision sequences⁵.
- There are examples of sentences which are grounded in the sense of ω -revision supervaluation but which are unstable in some revision sequence⁶.

This kind of investigation is also likely to shed more light on the actual, well-studied, Kripke-style and revision-theoretic proposals. The above-mentioned results suggest that ω -revision supervaluation can be viewed, in some sense, as intermediate between fixed-point and revision-theoretic semantics. Therefore it can play an important role also as a bridge in contrasting Kripke’s notion of groundedness with the revision-theoretic notion of stability.

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³Take the infinite sequence of sentences discussed in [9, pp. 210–211].

⁴Take Example 5.18 in [5, p. 401].

⁵See [2, p. 169] for the notion of near stability.

⁶Take Example 6B.9 in [2, p. 213].

Leibniz's Arithmetized Syllogistic: The Intensional Semantics

Vladimir Sotirov

Institute of Mathematics and Informatics, Sofia, Bulgaria
E-mail: vlsot@math.bas.bg

Abstract

Leibniz made many attempts to build an adequate model of syllogistic in arithmetic. Unfortunately all his models using natural relations between single integers (and not pairs of them) were unsuccessful. A working model of the intensional semantics preferred by Leibniz is presented here.

1 Introduction

The central idea of Leibniz's arithmetical models was to interpret *notions* by *integers* (being their *characteristic numbers*) and to translate syllogistic relations sAp ("all s are p ") and sIp ("some s are p ") into arithmetical relations between integers, using mainly *divisibility*. In such a way Leibniz hoped to realize his programme of calculating "the truth" and to promote his popular motto '*Calculemus!*'. Leibniz's favourite example was: if the number of *animal* were 2 and that of *rational* were 3 then the number of *man* being by definition a *rational animal* should be obtained by multiplication 3.2. Then the answer to the question "Is every man a rational being?" could be reduced to the fact that 6 is divisible by 3. Leibniz was acquainted with *two* equipollent semantics of the syllogistic propositions. Following his own expressions, the first semantics usually is called *extensional* and the second one is called *intensional*.

The example above is a proof that Leibniz was clinging to the intensional semantics of the universal affirmative (UA) propositions (the predicate is contained in the subject). Unfortunately there is no translation of a concrete particular affirmative (PA) proposition in the manuscripts. What is even worse, all attempts to use single integers turned out to be incorrect. In fact, Leibniz mixed both semantics: the interpretation of UA-propositions followed the intensional semantics while the interpretation of PA-propositions followed the extensional one. A detailed history of Leibniz's attempts including an analysis of the faulty procedures, the causes of the faults, some correct realizations of the primary Leibnizian idea together with a complete algebraic picture of Leibniz's logical systems from semi-lattices up to full Boolean algebras have been exposed in my previous paper [2] accessible from my web-site <http://www.math.bas.bg/~vlsot>.

2 Comparing Extensional and Intensional Semantics

Relevant to our task — an implementation of a natural and working translation of syllogistic into arithmetic — are Leibniz's manuscripts of April 1679 [1]. The example of 'man = rational animal' gives a criterion for UA-propositions: sAp is true when s is divisible by p . (The letters used for terms are the same as the ones used for their characteristic numbers.) In Leibniz's general notation: $s = xp$.

The interpretation of PA-propositions in Leibniz's manuscripts is less clear and more problematic. In his final variant Leibniz proposed sIp to be true when s , being multiplied by

another integer, is divisible by p . This is an arithmetical expression of Leibniz's reduction of PA-propositions to UA-propositions: sIp is true when s enhanced with an additional requisite x is p . In Leibniz's notation: $sx = yp$. However, if we take this rule literally, it will become trivially true because *any* integer s becomes divisible by *any other* integer p after multiplying it by a suitable integer, e.g., by p itself. That is why it is natural to complete the criterion for PA-propositions by the condition that the multiplier must be *less* than the number of the predicate. Then it is easy to prove that both criteria proposed by Leibniz can be formulated in a uniform manner: sAp is true when each divisor of p is also a divisor of s ; sIp is true when s and p have a common divisor greater than 1, or, $\text{gcd}(s, p) > 1$.

The equivalent formulation of the criteria using divisors of the characteristic numbers reveals the confusion of the two semantics promoted by Leibniz: sAp is true when the set of divisors of s includes the set of p (the intensional semantics) while sIp is true when s and p have a common part (the extensional semantics). As a result some syllogisms cease to be true, e.g., *Darii*. Both criteria can take their own places by obtaining in effect two adequate semantics. In the extensional arithmetical interpretation terms are evaluated by integers greater than 1; sAp is replaced with ' s is a divisor of b ', and sIp with ' $\text{gcd}(s, p) > 1$ '. If empty terms are admitted, they are evaluated by 1. For the intensional interpretation, an arbitrary integer $u > 1$ (a "universe") must be introduced and terms are evaluated by u 's arbitrary proper divisors (i.e., the divisors which are less than u); sAp is replaced by ' s is divisible by p ', and sIp by ' $\text{lcm}(s, p) < u$ ' (lcm denotes the least common multiple). If empty terms are admitted, they are evaluated by u . In both semantics the characteristic numbers do not admit multiple factors.

Considering integers s and p as sets of their (prime) factors and applying the usual symbols of set theory, sAp and sIp are true in the *extensional semantics* when $s \subseteq p$ and $s \cap p \neq \emptyset$ respectively (\emptyset is the empty set, the second relation corresponds to Leibniz's criterion for PA-propositions). For Aristotelian syllogistic all sets are supposed to be non-empty (and the characteristic numbers are different from 1). In the *intensional semantics*, sAp and sIp are true when $s \supseteq p$ and $s \cup p \neq u$ respectively (u is the universe; the first relation corresponds to Leibniz's criterion for UA-propositions). For Aristotelian syllogistic no set (number) is equal to u . The difference between both semantics is not big from a mathematical point of view because they are *mutually dual* (Leibniz calls them *inverse*). Namely, when the case is the one of sets, the intensional semantics will be obtained from the extensional one after replacing each set by its complement to u , \cap by \cup , \subseteq by \supseteq , \emptyset by u , and vice versa. When numbers are used, each number k has to be replaced by u/k , the expression 'is a divisor of' by the expression 'is divisible by', gcd by lcm , 1 by u , and vice versa.

The situation fundamentally changes when a configuration of the representing sets is drawn. Passing from extensional to intensional semantics all plausibility of the Leibniz circles (known as well as Euler circles and sometimes incorrectly named Venn diagrams) disappears. Our intuition loses the transparency and clarity of the overlaying and overlapping circles and leads us astray into the jungle of strange and artificial curves. Only an extremely inventive mind is able to draw a syllogism containing two particular propositions! In Figure 1 the rather elementary diagrams of sIp in both semantics should be compared: the black segment on the left side denotes that s and p have a common part and the black triangle on the right denotes that there is a part out of both of s and p .

The representation by circles in extensional semantics has the advantage to avoid the empty sets, so to say, automatically: it is not possible to draw a "null-circle". Respectively, the number corresponding to the empty set is 1 and it does not need to be introduced separately. It is constant for all models. The universe becomes necessary only when term negation is introduced into syllogistic. On the contrary, the universe u occurs in the interpretation of sIp

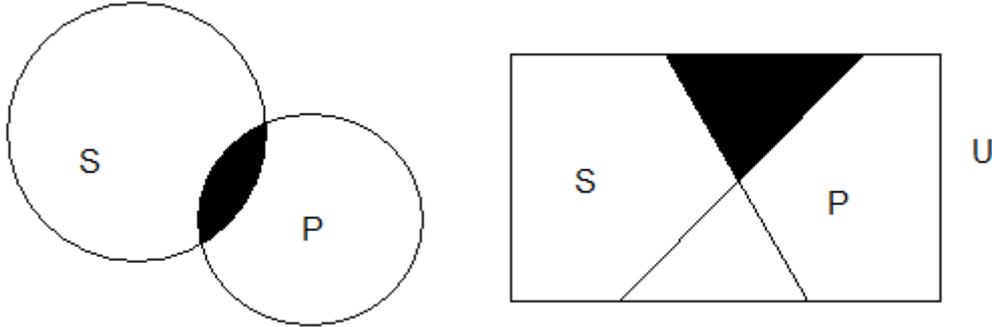


Figure 1: sIp in extensional semantics (*left*) and in intensional semantics (*right*)

in intensional semantics even when negation is not present. Moreover, the universe has to be changed if new elements are added to the model. Beside the intuitive and graphical obstacles intensional semantics represents some linguistic obscurities. Therefore it is not surprising that Leibniz invented a few perfect geometric representations following the extensional semantics and no one following the intensional semantics. We may conclude that Leibniz had correct geometric but wrong arithmetical intuition concerning syllogistic. The 20 year span between them is a possible explanation of this oddity. Obviously the intensional semantics is sophisticated if it presents a problem even to many contemporary authors.

3 The Intensional Arithmetical Semantics Working

In what follows we shall test the intensional arithmetical semantics on an extremely simple model, which nevertheless should be sufficient for the purposes of this work because it contains all Boolean term operations but not only the classical relations A and I . The example for our test is rather illustrative and does not claim adequacy to established taxonomy. In order to be able to compare the two semantics, the frame of notions is interpreted by characteristic numbers according to the extensional semantics and according to the intensional one (Fig. 2). Then the notions are treated as classes of *objects* in the first case and as classes of *properties* in the second case. The notion on the top is *animal*. From the extensional viewpoint it is the wider class of objects but according to the intensional viewpoint it has a minimal bundle of properties and none is specific. The class of *animals* is supposed to consist of *reptiles*, *mammals*, *birds*, and *insects* only. *Mammals* are subdivided into *dogs*, *mankind*, and *bats*. The easiest way to obtain the new characteristic numbers is to replace each number in Fig. 2 (left) by its reciprocal with respect to the “universe” ($u = 2.3 \dots 13$). The results are represented in Fig. 2 (right).

Now it would be straightforward to verify the UA-propositions by using simple division. For example, “Every man is a mammal” is true because the number $3.5.7.11.13$ is divisible by $7.11.13$. “Every bat is winged” is also true: $2.3.7.11.13$ is divisible by $2.3.7$. Let us check some PA-propositions. For example, let us answer the question “Are there winged mammals?”. We have to verify if there is a prime number between 2 and 13 that divides neither the number of *mammals* ($7.11.13$) nor the number of *winged* ($2.3.7$). Such a number is 5 and $2.3.7.11.13$

obviously corresponds to the class of *bats*, which appears below *mammals* and *winged* on the diagram. According to our scheme, the answer to the question “Are there winged reptiles?” is “no”, because 2.3.5.11.13 together with 2.3.7 contain all factors of u . Indeed, nothing is placed below *reptiles* and *winged*. Here it is important to note that Leibniz treated existence like *logical consistency*. As a result of this we have to assume that *flying mammals* should exist even if nobody has seen them. Their existence is logically possible because no property of *mammals* contradicts a property of *winged* according to our data. On the contrary, *winged reptiles* do not exist because they cannot logically exist as all *reptiles* are *wingless*.

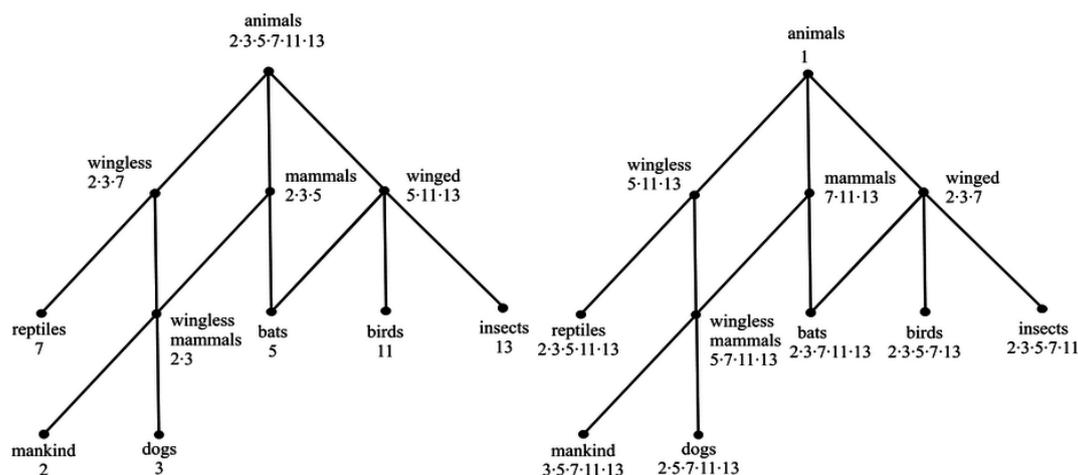


Figure 2: Extensional semantics (*left*) and intensional semantics (*right*)

Based on our example we can demonstrate the arithmetical translation of all Boolean operations with notions. If the question is disjunctive, e.g., “Which is the class of *men* and *dogs*?”, or in other words “What does unite *men* and *dogs*?”, the answer is: the class of the *wingless mammals* unites them because the $\gcd(3.5.7.11.13, 2.5.7.11.13)$ is 5.7.11.13 and the corresponding class is the nearest one placed above the *dogs* and *mankind*. Furthermore, the negation of *winged* is *wingless* (*apterouse*) and its number will be obtained by dividing u by 2.3.7 (for *winged*). Of course the result is 5.11.13 — the number of *wingless*.

Unfortunately, the full Boolean diagram contains $2^6 = 64$ vertices and is difficult to fit on a page. Therefore for the next consideration a simple algebra of $2^3 = 8$ elements (Fig. 3) would be useful. Take the three basic painting colours (as the *atoms* of the algebra): *yellow*, *red*, and *blue* and allocate the prime numbers 2, 3, and 5 to them respectively. Then the *pairs* of properties will represent the composed colours: *orange* (2.3), *green* (2.5), and *purple* (3.5). The “empty” colour *white* does not contain any other colour however it is included into each of the others. The combination of all three colours is *black* — the “universe”, and its number is 2.3.5. If a question “Does *orange* contain *red*?” is asked then the answer will be “Yes, because 2.3 is divisible by 3”. The answer to “Does *green* contain *red*?” will be “No, because 2.5 is not divisible by 3”. “Are *orange* and *purple* contained in any colour?” — “Yes, but only in *black*, because 2.3 and 3.5 together contain all factors.” “Do *orange* and *purple* contain a common colour?” — “Yes, because 2.3 and 3.5 have 3 as a common divisor and this colour is *red*.” “Which is the ‘negative’ colour of *blue*?” — 2.3.5 divided by 5 gives 2.3 and the answer is “*orange*”. &c.

Finally, let us return to the initial Leibniz criterion for *sIp*: there exist x and y such that $sx =$

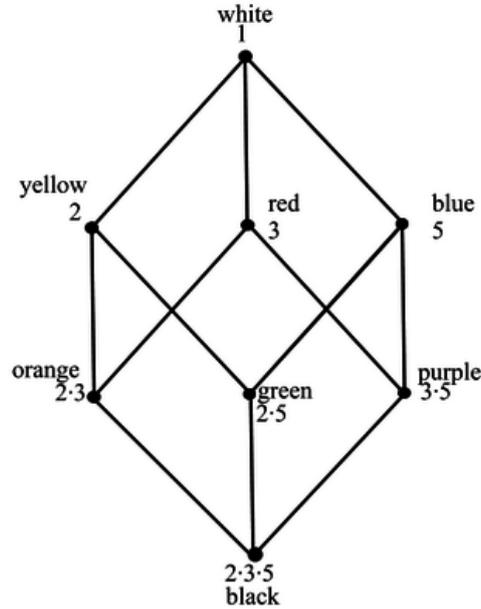


Figure 3: “Coloured” Boolean algebra

py. If $x < p$ and $y < s$ this requirement is equivalent to $\gcd(s, p) > 1$. The latter is equivalent to $\text{lcm}(s, p) < sp$. However we already know that the correct rule is $\text{lcm}(s, p) < u$. Indeed, the last example clearly shows the difference between trivial and appropriate calculations. For example, there is a colour containing both *red* (3) and *blue* (5) and this is *purple* (3.5). However we cannot consider this fact as a triviality because $\text{lcm}(3, 5) < 2.3.5$. The *composition* of two notions a and b (the conjunction of properties such as *rational animal*) always is ab in the intensional semantics (with the obvious limitation $cc = c$ for any factor c). The problem is in the existence of ab , i.e., in the truthfulness of aIb . The multiplication $3.2 = 6$ will not guarantee by itself that *rational animal* exists. Otherwise the fact that 6 is divisible by 3 with the same success would lead to the conclusion that “All flying men are flying” and “All square triangles are square” as well. In order to distinguish between the logically possible objects (e.g. *angels*) and the self-contradictory ones, a universe u is necessary to distinguish the case $ab < u$ from $ab = u$. This distinction has not been pursued by Leibniz.

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Rejection in Traditional and Modern Logics

Urszula Wybraniec-Skardowska

Opole University, Opole, Poland
E-mail: skardowska@gmail.com

Abstract

The paper comprises the genesis, development and generalization of the notion of the rejection of propositions whose idea was originated by Aristotle. It also outlines some important results of methodological research on that notion and the problem of decidability (saturation), in Łukasiewicz's sense, of deductive systems formalized twofold: both as systems of acceptance and as systems of rejection (refutation systems).

1 Introduction

Formal logic is divided into traditional and modern; the beginning of the second accrues in the years around the mid-nineteenth century. These two disciplines are different in genesis, scope and language.

The European logic as a science arose in 4th c. BC in ancient Greece; it also appeared earlier in ancient China and India. The traditional logic arose in the times of the so called „Athenian democracy” out of the needs of Greek citizens who were actively participating in public and political life. Aristotle is thought to be the creator of logic. Aristotle has great merits in analysing language, but first of all he is recognized as the creator of the first formal–logic system, deductive system, the so called Aristotle's syllogistic, which together with the theory of immediate reasoning (square of opposition, conversion, obversion, contraposition, inversion) is treated as traditional logic in a narrower meaning. It is the logic of names.

Aristotle's syllogistic is the first system about the forms of correct reasoning. Not long time ago syllogistic was the core of *classical logic*, an important fragment of formal logic, which is the basic field of the contemporary logic.

In a little later ancient times another logical-deductive theory was two-valued sentential logic created in Greece in the 3rd c. BC by Stoics; it is also included in traditional logic (in a broader sense). The logic of Aristotle and the logic of Stoics were the bases of teaching in Ancient Times – until the 6th c. AD; and later in Medieval Times (6th c. AD – 14th c. AD). The traditional logic in The Middle Ages was included in *trivium* – the base of teaching subjects (next to rhetoric and grammar).

2 The Genesis of the Notion of Rejection

The idea of rejection of some sentences on the basis of others was originated by Aristotle, who in his systematic investigations regarding syllogistic forms not only proves the truth but also rejects the false (invalid, erroneous) either rejecting false ones by means of examples or using the method of rejection of some false syllogistic forms on the basis of already earlier recognized as

false (invalid, erroneous). The method consisted in reducing some false syllogistic forms to other ones, the erroneousness of which was already shown.

This Aristotle's idea and method of rejecting sentences was found by Jan Łukasiewicz during his long studies over Aristotle's syllogistic. He discusses it in his pre-war article 'On Aristotle's syllogistic' in Polish (1939) which is the summary of his lecture delivered at the meeting of Polish Academy of Sciences (Polska Akademia Umiejętności) and presenting his results from the entirely finished monograph on Aristotle's syllogistic, which later burnt during the fights in Warsaw; Łukasiewicz reconstructed it after the war and published in English in Oxford (1951), it contained an entailed elaboration of the results presented in the work prepared before the war.

The method of rejecting sentences always functioned in empirical sciences in connection with the procedure of invalidating hypotheses and we can assume that it was also known to Stoics: within the five 'unprovable', these accepted without any proof rules of concluding of their logic of sentences – hypothetical syllogistic rules, formulated probably by Chrisippus, because as the second one appears the rule of deducting called in Latin *Modus Tollendo Tollens*, shorter *Modus Tollens*: If p , then q ; not- q , so not- p .

The Aristotle's idea of rejecting some expressions on the base of others was not properly understood by logicians and mathematicians, especially those convinced that rejecting closed sentences of the language of deductive system can be always replaced by introducing negations of such sentences into such system.

The notion of rejecting was introduced into formal logic by Łukasiewicz already in (1921) in his paper in Polish „Logika dwuwartościowa” („Two-valued logic”), but proper understanding Aristotle's ideas and introducing the notion *rejecting* into formal researches of logical deductive systems, is not revealed until his researches of Aristotle's syllogistic (1939, 1951), and later in his metalogical studies of some propositional calculi (1952, 1953). In that research Łukasiewicz makes use of the Aristotle's idea of rejection to complete syntactical characterisation of deductive systems using an axiomatic method of rejection introduced by him into formal logic in this paper (1939) and then, in the monograph (1951). In the article (1939) we read (cf. Ślupecki, Bryll, Wybraniec-Skardowska 1972, p.76):

“No syllogism rejected by Aristotle follows from the above axioms and rules of inference but they do not entail, too, erroneousness of rejected syllogism. In order to solve this problem the system should be extended in a way. The author has chosen here the following way which has not been used in deductive systems so far. Besides *asserted* theses in the system he names propositions which are *rejected* in the system and rejects axiomatically some of them and reduces the others to propositions which already rejected by means of special *rules of rejection*”.

As a rule of rejection (refutation), corresponding to the rule of detachment by assertion, Łukasiewicz adopts (1939, 1951) the rule of rejection by detachment which was anticipated by Aristotle, while as the rule of rejection (refutation) corresponding to the rule of substitution for assertion he adopts the rule of rejection by substitution which was unknown to Aristotle. Łukasiewicz uses with it in his researches the axiomatic method of rejecting.

The notion of rejected proposition of a system, e.g. Aristotle's syllogistic, is inductively defined by Łukasiewicz in the following way:

1. Some propositional expressions formulated in the language of a certain system have been classified as *rejected axioms*.
2. The other expressions *are rejected* by means of expressions which are already rejected as well as by means of the rules of rejection:
 - a. the rule of rejection by substitution: any expressions is rejected if one of its substitutions is a rejected expression,

- b. the rule of rejection by detachment: if there exists such a rejected expression β that the conditional sentence build of α as the antecedent and β as the consequent is the theses of the system under consideration (is asserted) then α is rejected expression.

The rejected rules intuitively lead from some false formulas to false ones of this system (while inference rules intuitively lead from true formulas to true formulas of this system). So, if rejected axioms of this system are false all rejected its formulas are false.

3. Biaspectual Syntactic Characterization of Deductive Systems

Introducing into contemporary formal logic the notion ‘*rejection*’, as an operation opposed to Frege ‘*assertion*’, Łukasiewicz introduced at the same time biaspectual axiomatic characterisation of some deductive systems, including Aristotle’s syllogistic. The biaspectual syntactic characterization of deductive systems analysed by him (1939, 1951, 1952, 1953) consists in providing both:

- (*) the axioms and inference rules for a given deductive system, and
- (**) the rejected axioms (treated as false or non-accepted formulas of this system) and rejection (refutation) rules of this system.

Such a syntactic formalization of some propositional calculi was also, probably independently, introduced by Rudolf Carnap (1942, 1943).

Łukasiewicz in his studies on Aristotle’s syllogistic, as well of systems of propositional calculi (1952, 1953), makes use of the notion of *decidability*, also termed *saturation* of a deductive system, and the notion *consistent system* in the meaning different from the one stock in logic. He bases these notions on the notion of rejected proposition, originated by himself. These notions were formulated by Łukasiewicz by means of exemplification, without providing any clear definition, however, as indicated by contexts, the meaning intended by him – and accepted by his disciple Jerzy Śłupecki – was as follows:

3. The system is *decidable* if every of its expressions which is not a thesis is rejected on the basis of finite number of axiomatically rejected expressions,
4. The system is *consistent* if none of its thesis is rejected.

The biaspectual formalization of Aristotle’s syllogistic done by Łukasiewicz, comprising – beside “ordinary” axioms and inference rules – rejected axioms and the rules of rejection a. and b., made it impossible for him to obtain a satisfactory solution of the question of decidability of this system. The solution was found by his disciple and a continuator of his research Jerzy Śłupecki. Having added to the syllogistic system a new rule of rejection, specific to the syllogistic, Śłupecki proved in 1938 that the syllogistic system, enriched this way, is both decidable and consistent. The result achieved by Śłupecki – in words of Łukasiewicz (1939) – “organically united with the researches of the author ... the author regards as the most significant discovery made in the field of syllogistic since Aristotle”.

4 Results on Ł-decidability

Researches on rejecting expressions and those started by Łukasiewicz on decidability of deductive systems in Łukasiewicz’s sense, called by Śłupecki *Ł-decidability*, were propagated in Poland by Śłupecki in the Wrocław-Opole environment (the region from which comes the author of this paper) in 1960s and 1970s. Such researches consist in biaspectual characteristic of

deductive system as: 1) *assertive system* $Cn^+(A^+, R^+)$, i.e. system of thesis, in which the operation of consequences Cn^+ is infallible: on the base of accepted (recognized as true) axioms A^+ of system, with use of infallible rules R^+ leads to sentences recognized as true, and 2) disjoin with it *refutation system* $Cn^-(A^-, R^-)$, in which the operation of consequences Cn^- from not accepted (rejected, false) axioms A^- of the system, using the rules of rejecting R^- leads to not accepted (rejected, false) sentences of the system. The system is *L-decidable (saturated)*, if both of these disjoint systems give in total the set S of all well-formed expressions of the system, i.e.:

$$Cn^+(A^+, R^+) \cup Cn^-(A^-, R^-) = S,$$

which means that the set of non-thesis of the system is the set of its rejected expressions, i.e. it gives a refutation system.

Using the method of rejecting expressions and studying Ł-decidability of logic systems became not only in Poland, but also abroad a popular methodological procedure.

The following paper is intended (cf. Wybraniec-Skardowska (2005), Wybraniec-Skardowska, Waldmajer (2010)): 1. to remind and discuss the above achievements of Łukasiewicz and Słupecki, 2. to present a general overview of results of research regarding the problem of Ł-decidability, 3. to outline certain studies of generalization of the notion of rejected expression in the form of a function of a consequence of rejection or a dual consequence which contributed greatly to formulation of the theory of rejected propositions.

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Logic and Ontology in Ancient India^{*}

Piedad Yuste¹ and Ángel Garrido²

¹Department of Philosophy (UNED); E-mail: pyuste@fsof.uned.es

²Department of Fundamental Mathematics (UNED); E-mail: agarrido@mat.uned.es

1 The origin of Logic in Ancient India

In ancient India epistemology and logic emerged in a context of dialectical debates and clashes of various kinds (science, religion or metaphysics), in which it was not always intended to reach the truth, but sometimes, the primary objective was to defeat the opponent, whatever her thesis. The structure of these debates was ordered according to the question/ response pattern and did not rule out the use of fallacies and linguistic tricks to destroy the arguments of the opponent. In ancient India, the sages of Nyāya School (III-II B.C.E.) created a model of inference from which assure the veracity of assertions, while the remaining schools adapted this same methodology regardless of their own ontological conception of nature. After several centuries, the Yogācāra School, one of the philosophical currents further away from Nyāya, managed to improve and complete this reasoning model and take the leap from an intensional logic to another of extensional character. We asked if it had to do with their different view of the world or it was simply a natural evolution towards a better understanding of the problem.

In this paper we will try to analyze the different ontological conceptions of both schools by linking them to their corresponding logical postulates.

2 The logic of the Nyāya School

In India, the Nyāya School (3rd-2nd centuries B.C.E.) was especially devoted to studying reasoning and its laws, while the rest of the schools took advantage of its methods and results to defend their respective philosophical and religious doctrines.

The term “nyāya” means “right manner” (Presendanz 2009, 30). This school, merged with the Vaiśeṣika school (10th century C.E.), is one of the great schools of thought of ancient India. Because this school accepts the authority of the Vedas, it is considered an orthodox system or *darśana*: they believe in the existence of an Atman or individual spirit, in contrast to Buddhism, which denies the possibility of the self, although both doctrines predict a phase of mystical illumination or liberation of the soul and its dissolution in the whole.

The Nyāya - Vaiśeṣika doctrine is, therefore, a dualist doctrine; in a way, its epistemology is reminiscent of Aristotelian epistemology, of an essentialist and realist nature; although, on the other hand it defends the idea that nature is made up of atoms. These factors, essentialism, dualism, and realism, are the ones that are categorically opposed to Buddhism and that determine its logical positioning. For centuries, the Nyāyayikas kept up a fierce controversy with the Buddhists regarding these issues, with the leadership in developing the theory of knowledge and of logic alternating between one group and the other. These discussions reached their high point in the 10th and 11th centuries of our era, when Udayana (most certainly the sharpest philosopher of the Nyāya-Vaiśeṣika school) wrote his treatises and managed to definitively defeat his erudite adversaries. From that moment onwards, the Buddhist movement disappeared, little by little, from

^{*} This communication is carried out under the research project directed by Quintín Racionero (†) and Cristina de Peretti: “El papel de las controversias en la producción de las prácticas teóricas y en el fortalecimiento de la sociedad civil [The role of controversies in the theoretical practices production and strengthening civil society]” (MICINN: FFI 2011-29623).

the Indian subcontinent and moved toward China, Tibet, and other southeastern Asian countries, where it managed to take root. After this, there was a movement to renew the Nyāya theses, carried out by Gaṅgeśa (1350 C.E.), founder of the Navya- Nyāya school.

The teachings of the Nyāya school were compiled by Akṣapāda Gautama (2nd century C.E.) in the *Nyāyasūtra* (NS) and, as was habitual among Indian thinkers, reviewed and discussed on numerous occasions. This work, of an epistemological nature and written in the form of aphorisms, describes the guidelines that philosophical debate (*vādayutti*) must follow. The dialectical resources used in the colloquia are the same that many Greek philosophers, such as the Eleatics, Megarics, and Sophists and, later, the ancient Sceptics, applied in their day: reduction to absurdity (*prasanga*), return to the infinite (*anavastha*), and the vicious circle (*cakra*). Akṣapāda distinguished among three classes of dialogue, depending on whether the intention was to reach the truth (*vāda*), demonstrate the falseness of an opinion without proposing another different opinion (*viṭaṇḍā*), or, simply, win the argument without reaching any positive conclusion or advance in knowledge (*jalpa*).

In the *Nyāyasūtra*, the four means for obtaining knowledge or *pramāna* that had already been announced in the medical text *Carakasamhitā* are analyzed: perception (*pratyakṣa*), inference (*anumāna*), comparison or analogy with similar phenomena (*upamāna*), and trust in the testimony or authority of others (*āgama*). Let us stop for a moment and look at *inference*: the element upon which Indian logic pivots and which is carried out, according to Akṣapāda, in five steps (NS I.I.32). The author proposes several examples to support this:

- Proposition (*pratijñā*): *This hill is fiery,*
- Reason or justification (*hetu*): *because it is smoky,*
- Example (*udāharana*): *Whatever is smoky is fiery, such as a kitchen.*
- Applying the example. (*upanaya*): *So is this hill (smoky).*
- Conclusion (*niḡamana*): *Therefore this hill is fiery.*

Here, the subject (*pakṣa*) is *mountain*, the justification (*hetu*) is *smoke*, and the predicate (*sādhya*) is *fire*. The relation that ties *hetu* and *sādhya* together is invariable: one property cannot be conceived of without the other. In this specific case, there is a concordance of cause/effect (*pūrvavat*).

In the opinion of the Nyāyayikas, in order for the inference to be valid, there must be a universal connection between the reason (*hetu*) and the predicate (*sādhya*), or a total independence between the reason and the absence of the predicate (NS V.I.3).

As we can see in these examples:

Sound is not eternal, because it is produced, like a pot.

Space is permanent, because it is not produced, like time.

The second proposition plays with the incompatibility of the qualities “permanent” and “produced.” In contrast, the statement:

Sound is eternal, because it is incorporeal, like the sky,

is incorrect because there is no universal relationship between the two properties “incorporeal” and “eternal”; that is, there are exceptions, as happens (as they say) in the case of the intellect, which they see as impermanent and incorporeal. As a result, it can be stated that *everything that is eternal is incorporeal*, but the opposite is not the case. This relation is defined quite well in the following fragments:

A homogeneous example is a familiar instance which is known to possess the property to be established and which implies that this property is invariably contained in the reason given.

A heterogeneous example is a familiar instance which is known to be devoid of the property to be established and which implies that the absence of this property is invariably rejected in the reason given (NS I.XI.36).

The successive authors of the *Nyāyasūtra* were no more explicit when it came to presenting the requirements that the terms that intervene in inference must fulfill; they only showed a collection of arguments (that they called futile) based on which a supposed opponent could try to neutralize well-constructed reasonings (*NS* V.I.4 ff.).

Each pair of properties chosen denotes the same elements; that is, they are co-present in the same locus, but this is not the reason for inverting the function that they carry out in the sentence:

Sound is not eternal, because it is a product, like a pot,

because one of these properties (*eternity*) is less well-known or not perceived, as:

Inference gives us the knowledge of an unperceived object through the knowledge of an object which is perceived (*NS* II.I.110).

And in case of doubt, the *Nyāyasūtra* anticipates contemplating difference, too:

In this case we must bear in mind that we cannot ascertain the true nature of a thing unless we weigh it in respect of its homogeneity with as well as heterogeneity from other things (*NS* V.I.15).

As could be expected, this outline of reasoning underwent different revisions and adjustments from the Nyāyayika commentators (Vātsyāyana, Uddoyotakara, Vācaspati, and Udayana) and the Buddhist commentators (Asanga, Vasubandhu, Dīnāga, Dharmakīrti, Dharmottara, and Śāntarakṣita). The purpose was to define the conditions that ensured its consistency.

3 Ontological aspects of Nyāya logic

The realism of this school determines its criterion of truth: « Knowledge is true if it agrees with reality ». This principle is very similar to the one we can find in Plato (*Sophist* 263c) and, later, in Aristotle (*Metaphysics* III, 1011b). This realism involves postulating, implicitly, three logical principles: *Non-contradiction* [$\neg(A \& \neg A)$], *Identity* [$A = \neg(\neg A)$] and *Excluded Middle* [$A \vee \neg A$]; admitting, as a result, the concept of bivalence (*virodha*) and restricting the universe of discourse to two unique values of truth (Bandyopadhyay 1988, 225):

In the case of mutual contradiction there is no third alternative. There is also no identity of contradictories, for the contradiction is apparent on the very face of assertions .

The representatives of this school also examined the relation of contrariness, which admits a third possibility, as happens when we predicate different qualities of a single subject: some or none may be true. Here the principle of bivalence does not apply.

The problem of the change that beings undergo in nature, the transition from being to not being [origin of the *sorites* paradox formulated by Eubulides of Miletus and to which Aristotle alludes when he states that: « there is a more and a less in the nature of beings » (*Metaphysics* IV 4 1009a)], was interpreted quite differently by the Indian philosophers.

The Nyāyayikas explained it according to their atomist hypothesis: the development of one organism follows the decadence of another and the decomposition of its parts (*NS* III. II.92). Thus:

The growth of a thing is the increase of its parts while the decay is the decrease of them (*NS* III. II.88).

This expression is conceived in a substantialist perspective:

The Nyāyayika says that there is certainly a substance apart from its qualities and a whole apart from its parts (*NS* IV. I.35).

In the opinion of Debiprasad Chattopadhyaya (1978, 126), many of the ideas that the Indian philosophers held about change come from Ayurvedic medicine, as we can read in a fragment of the *Carakasamhitā*:

Nothing about the body remains the same. Everything in it is in a state of ceaseless change. Although in fact the body is produced anew every moment, the similarity between the old body and the new body gives the apparent impression of the persistence of the same body.

The observation of the process of nutrition, the constant absorption and elimination of substances, urged doctors to conceive the human body as an organism subjected to a continuous and absolute renovation of its constituent elements. This led to a generalization of this idea for all other beings that make up the universe.

The Nyāyayikas conceived the possibility that a material substratum exists that underlies change, as is indicated in *NS III.II.88*:

We never find an instance in which a thing decays without leaving any connecting link for another thing which grows in its place.

This notion could have been taken from the Ayurvedic tradition, according to which everything in nature is made up of matter in five forms (water, air, fire, earth, and ākāśa) and these forms are what possess specific properties that they later transfer to their compounds, whether they be organic or mineral. Similarly, there is a correlation between the quantity of constituent elements of a body and the qualities that it manifests, as Udayana (10th century C.E.) declared, according to whom:

From the loss or gain in the body it is indisputably proved that the old body ceases to be and is replaced by a new body (Bandyopadhyay 1988, 134).

These thinkers, then, do not seem to be concerned about obstacles like the ones that Eubulides of Miletus considered. Centuries later, the last representatives of the Nyāya-Vaiśeṣika school tried to reconcile the notion of change, exclusive to matter, with the belief in an immutable spirit.

4 General features of Buddhism

The teachings that Buddha Buda Śākyamuni (6th century B.C.E) imparted to his disciples were, as Stcherbatsky comments, of quite different content and difficulty: the teachings related to the world of the senses produced the Hīnāyana (or Small vehicle) doctrine, while those of an ontological nature created the seed of the Māhāyana (Large vehicle) school. This last, in turn, diversified into two schools: the Mādhyamaka (skeptical) school and the Yogācāra (idealist) school. Afterward, between the 5th and 10th centuries C.E., we find a less speculative Buddhism: Tantrism.

All of these philosophical schools coincide in considering the universe and the beings that make it up to be impermanent and empty; that is, lacking a nature of their own. This means that the objects that we perceive are only aggregates of substances that are trivial and in a continuous process of change. There is neither an essence nor a thing-in-itself, but only the instantaneous and interdependent phenomenon. Similarly, they reject the existence of a self or individual cognizant subject: this is only a succession of states of consciousness, so that there is no stable and permanent element of identity that supports it.

The earliest Buddhism accepted the substantiality of the *dharmas*, the final elements of which make up the physical and mental world, so that this doctrine possessed a certain pluralist and realist character. On the contrary, Māhāyana Buddhism drifted toward non-dualist conceptions of reality, as did the respective followers of the Mādhyamaka and Yogācāra schools, often opposed to one another due to the radical skepticism of the first school.

5 Buddhist idealism: The Yogācāra school

In contrast to Mādhyamaka philosophy, the Yogācāra idealist school developed the ideas of the Māhāyana without descending into skepticism. The most important representatives of this school

of thought were the brothers Asanga and Vasubandhu, Vasubandhu's disciples Diñnāga, and his successor Dharmakīrti. These last representatives had a hard time conjugating the idealism of the school with its dualist vision of the world, according to which there are two kinds of reality: the reality of the individual external object and the image that the mind develops of this object. They all based themselves on the Nyāya theory of knowledge, perfecting and trying to harmonize it with the Buddhist theses.

The logicians of this school stated that perception and inference are the only ways of achieving knowledge, as the other three means are included in these. Similarly, they reviewed the formal aspects of the five-step model of inference and reduced it to three (*trairūpya*), considering the others (application of the example and conclusion) to be mere repetitions of the first three.

Vasubandhu (5th century C.E.) systematized the Nyāya concept of inference: this concept is valid if there is a general relationship between the property designated by reason (*hetu*) and the property that one intends to induce (*sādhya*), so that this last property never appears without the first property; but this does not signify authentic progress with respect to the Nyāyayika formulation. Vasubandhu's disciple Diñnāga (480-540 C.E.) added the notion of necessity to the *principle of invariable concomitance* (*vyāpti*): « all things denoted by the middle term (*hetu*) must be homogeneous with things denoted by the major term (*sādhya*); none of the things heterogeneous from the major term must be a thing denoted by the middle term» (Keith 2012, 134). That is, the property that is stated about the subject (*pakṣa*) must appear in its homologues (*sapakṣa*) and never in its heterologues (*vipakṣa*). What does this mean? Diñnāga considers entities that, fulfilling the condition established by reason or the mark (*hetu*), also possess the property announced in the predicate (*sādhya*) to be similar entities; different entities are things that do not contain the attributes expressed in the predicate and the reason.

Diñnāga constructs his theory with his eyes on a specific kind of inference: the inference in which the mark designates only part of the objects denoted by the predicate. It is a matter of converting a rudimentary intensional logic to another more advanced logic that we call extensional, as in the following example:

Words are not eternal, because they occur voluntarily.

The mark, «to be produced voluntarily» denotes the elements of a class contained in another of greater extension: the class of «non-eternal beings».

In addition, Diñnāga introduced the concept of double negation (*apoha*), closely linked to Buddhist ontology, arguing that we do not know with certainty what a thing is, but we do know what it is not. With these tools, he managed to distinguish new ways of inference depending on how the reason is related to the respective homologues and heterologues.

The mark can be present in some, in all, or in none of the homologues, and absent in some, or all, or none of the heterologues. If we combine these possibilities, we obtain nine forms of inference predicted by Diñnāga, of which only two are correct:

- When the mark is present in that which is similar and absent in that which is different.
- Partial presence of the mark in that which is similar and absence in that which is different.

Dharmakīrti (in his *Nyāyabindu*) added the Sanskrit term *eva* and its negation to further refine the characteristics of the correct inference. The mark «it must exist (*wholly*) in what is to be inferred; it must exist in things *only* which are homogeneous with the major term; and it must not exist (*never*) in things heterogeneous with the major term».

This term acts as a quantifier, which means that it comes closer to Aristotelian syllogistics. Thus, we interpret the two cases of valid inference mentioned by Diñnāga as follows:

- The mark is predicated of all the homologues and of none of the heterologues [AE].
- The mark is predicated of some homologues and of none of the heterologues [IE].

Buddhist logicians used pairs of principles similar to those called A, I and E in European scholasticism, combining them as follows: AA, AE, AI, EA, EE, EI, IA, IE, II. The first element of the pair designates the homologous objects and the second, the heterologous objects. Only the second and eighth types are correct. The Indians managed to formalize their logic in this way, subjecting it to a set of well-defined principles and rules.

Fallacies arise when any of the conditions expressed previously are violated.

6 Ontological aspects of Buddhist logic

Thinkers of the Yogācāra School professed a pragmatic kind of idealism which did not contemplate either the figure of an eternal, creator god or an individual self. Given that it is impossible to know the authentic reality of phenomena due to their continuous evanescence, we must make do with the data that experience provides: «Right knowledge is knowledge not contradicted (by experience)» (Stcherbatsky 1930, 4). On the other hand, «right knowledge is efficient knowledge» (*Ibid.*, 39); that is, if it is capable of guiding people's actions within this illusory, conventional world that they themselves have organized.

Thus, they understand that logic is a very useful instrument when it comes to refuting their ideological adversaries and convincing those who are undecided, especially in matters of religion. In this sense, the Yogācārins act in a similar fashion to their realist rivals, although, for the Buddhists, the criterion of truth is linked to the absence of contradiction [$A = \neg(\neg A)$] and not to checking whether or not the mental object corresponds to the perceptible object (Nyāyayikas).

Diñnāga and Dharmakīrti introduced the notion of the point-instant (*kṣaṇa*) to explain the ultimate, changing reality of nature: ephemeral moments of existence only perceived in a first contact with the phenomenon. Immediately afterwards, the mind organizes and unifies the data supplied by the senses and uses it to compose a static image of things. Based on this image and by applying the laws of knowledge –contradiction, identity, and causality– inferences are modeled and judgments constructed (Stcherbatsky 1930, 37, 127):

Consequently real is only the (i.e., the unique point of efficiency, the thing in itself), not the constructed object of imagination.

Each point-instant arises dependent on other point-instants that precede and condition it, so that:

Nothing simple comes from single. From a totality everything arises .

Thus, this underlines the holistic, dynamic behavior of the universe. Cause and effect can never be simultaneous, as the realists defend, but rather the first necessarily comes before the second. A necessary connection means a dependent existence, and «there is no other possible dependent existence (than these two, the condition of being the effect of something, and the condition of being existentially identical with something)» (Stcherbatsky 1930, 75).

Yogācāra ontology rejects contradiction, as the phenomena of nature follow one another gradually, excluding one another mutually. On the mental level, they declare that everything that is contradictory is not real but conventional, produced by language. Regarding the possibility of there being a third alternative, they do not seem to be very sure, as (*Ibid.*, 210, 216):

A fact which points to an indefinite position of the subject between two mutually exclusive attributes is a source of doubt.

Although their pragmatism induces them to admit bivalence in the sphere of discourse:

The mutual exclusion of two facts means that the absence (of the one is equivalent to the presence of the other).

From the ontological perspective of Buddhism, the step from being to not being involves the co-existence of opposite elements, so that it does not make sense to discuss the sorites paradox, as

can be inferred from the words written by a commentator and critic of Buddhist doctrine, Vācaspatimiśra (900-980 C.E.):

[...] the objects are different, because it would be a contradiction to admit that the same single object resides in a former and in a following space-time. (A difference in space-time is a difference in substance).

In this fragment (*Ibid.*, 1930, 282), the author does not refer explicitly to the change that bodies experience over time (the example that he proposes refers to a mineral), but from his reasoning we can draw the conclusion that just as a ruby is different in each time-space instant, this is even more the case for any being that transforms and evolves. On his part, Dharmakīrti created a reasoning to explain the succession of naturally opposed phenomena, such as light and darkness, the sensation of cold and that of heat ((Stcherbatsky 1930, 189). The movement from light to darkness involves the existence of a neutral element that separates the two sequences. As a result, the mystery about the *sorites* reasoning, which has caused so many discussions in the western world, does not seem to worry Buddhist logicians; in fact, given their specific conception of nature, it would not even have occurred to them. The scale of grays of which Aristotle spoke (*Metaphysics* IV.7, 1011b) would only make up an aggregate of evanescent points, independent of one another and following one another in time.

Despite their discrepancies, Buddhist logicians used the same language and the same rules of reasoning as did their realist rivals: each subject and predicate possesses a single, universal meaning, even though, for the Buddhists, these are only devices, tacit agreements, not correlates of reality. In contrast to their close associates the Madhyāmikas, the Yogācārins did not resort to the notion of the *middle ground* but rather to the principle of bivalence, although this leads to the rupture between logic and ontology. We wonder whether philosophers as lucid as Dinnāga and Dharmakīrti were aware of this.

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Argumentation Logic as an Anhomomorphic Logic

Fay Dowker¹, Antonis Kakas², and Francesca Toni³

¹ Blackett Laboratory, Imperial College London, UK
E-mail: f.dowker@imperial.ac.uk

² Department of Computer Science, University of Cyprus, Cyprus
E-mail: antonis@ucy.ac.cy

³ Department of Computing, Imperial College London, UK
E-mail: f.toni@imperial.ac.uk

Abstract

Recent work proposed Anhomomorphic Logic, characterized in algebraic terms via answering maps on a Boolean algebra of propositions, as a logical framework appropriate for physics, including both classical and quantum theory. In this paper we study how another type of logic, called Argumentation Logic, generalizing classical logic to accommodate various forms of commonsense reasoning, exhibits properties of Anhomomorphic Logic.

1 Introduction

Argumentation Logic (AL) captures entailment in classical logic as well as forms of defeasible commonsense reasoning [4, 5, 6]. AL-entailment is defined through argumentation, in terms of the least fixed points of ‘acceptability’ and ‘non-acceptability’ operators, for a given (possibly empty) theory in a propositional language equipped with an underlying notion of ‘direct derivation’. This is some fragment of derivability in classical logic, but excluding *reductio ad absurdum*.

Anhomomorphic Logic (AnhomL) was developed to deal with the modified rules of inference that may be needed for quantum physics [7, 10, 1]. It is framed in terms of a Boolean algebra of propositions about a system and a set of allowed answering maps. A *Scheme* for AnhomL is a set of conditions the allowed answering maps must satisfy which define the type of inferences allowed in the logic. It remains an open question which Scheme – if any – will successfully account for the physics we know.

The central feature of both AnhomL and AL is that they “tolerate contradiction” without collapsing into triviality, so it is interesting to explore the possible connections between them. We make a start in this direction by defining an answering map, χ^T , corresponding to AL-entailment, and show that χ^T has algebraic properties that can be compared to those of answering maps in AnhomL.

The paper is organised as follows: we first briefly recall essentials of AnhomL (in section 2) and AL (in section 3); we then define and study χ^T (in section 4), illustrating it with the 3-slit example from quantum physics (in section 5), and conclude in section 6.

2 Anhomomorphic Logic

Anhomomorphic Logic [7, 10, 1] pertains to a collection, \mathcal{U} , of propositions about a (physical) system. \mathcal{U} is a Boolean algebra and closed under the propositional logical connectives of \wedge, \vee, \neg . Elements of \mathcal{U} are also referred to as *events*. This structure naturally arises in physics where there is an underlying set, Ω , of spacetime histories of the system such that elements of \mathcal{U} are subsets of Ω . Then \wedge, \vee, \neg are defined in the canonical way through the intersection, union and complement set operations. A possible world is an answering map, $\phi : \mathcal{U} \rightarrow \mathcal{L}_2 = \{0, 1\}$ (called a *co-event* in the literature, since ϕ maps events to a set of scalars), and if $\phi(A) = 1$ (respectively $\phi(A) = 0$) we say A is *affirmed* (respectively *denied*) by the world ϕ . A *Scheme* is a set of conditions that an *allowed* answering map must satisfy. We

define here some conditions on ϕ that may or may not hold within any given Scheme. First it is useful to define the logical connective ‘exclusive or’, \oplus , on events in \mathcal{U} : $A \oplus B := (A \vee B) \wedge \neg(A \wedge B)$. Then ϕ is *additive* if $\phi(A \oplus B) = \phi(A) + \phi(B)$ for all A and B in \mathcal{U} . ϕ is *multiplicative* if $\phi(A \wedge B) = \phi(A)\phi(B)$ for all A and B in \mathcal{U} . ϕ is a *homomorphism* if $\phi \neq 0$ (namely there is some A such that $\phi(A) \neq 0$) and is both additive and multiplicative.

The rules of logical inference are determined by the Scheme conditions. For example, if the Scheme condition is that ϕ is a homomorphism then the rules of inference are those of classical logic [7]. Hence the term *Anhomomorphic Logic* to denote the framework: in a Scheme with less restrictive conditions, allowed worlds will not be homomorphisms and the rules of inference about the world will not be classical. The Multiplicative Scheme (MS) for AnhomL [7] is defined by the conditions that ϕ is multiplicative and not constant, together with a minimality condition and a dynamical condition known as preclusion (see the triple slit example below in section 5). In the MS scheme the rule of inference $(\phi(A) = 1 \Rightarrow \phi(\neg A) = 0) \forall A \in \mathcal{U}$ holds but the rule of inference $(\phi(A) = 0 \Rightarrow \phi(\neg A) = 1) \forall A \in \mathcal{U}$ does not.

3 Argumentation Logic

Argumentation Logic (AL) [4, 5, 6] defines entailment in terms of notions of acceptability and non-acceptability of arguments, seen as sets of propositional formulae in a given language \mathcal{L} , equipped with a notion of *direct derivation* based on a subset of standard inference rules in Natural Deduction (see appendix A). When a formula A in \mathcal{L} is directly derived (using the chosen inference rules) from a theory T in \mathcal{L} , we write $T \vdash_{DD} A$. Whereas in [4, 5, 6] the chosen inference rules, and therefore \vdash_{DD} , are fixed, and amounting to all inference rules in appendix A except RA (Reduction ad Absurdum), in this paper we do not commit to a specific \vdash_{DD} , and leave it instead as a parameter for AL, while imposing that it does not include RA. We say that a theory T in \mathcal{L} is *directly inconsistent* if $T \vdash_{DD} \perp$, where \perp stands for inconsistency and amounts to $A \wedge \neg A$ for any A in \mathcal{L} . We say that a theory is *directly consistent* if it is not directly inconsistent. Throughout this paper we will assume as given a directly consistent theory T .

The notions of (non-)acceptability are defined in terms of notions of *attack* and *defence* amongst arguments (i.e. T extended by sets of formulae), as follows, for Δ, Γ sets of formulae in \mathcal{L} :

- argument $a = T \cup \Delta$ attacks argument $b = T \cup \Gamma$, with $\Gamma \neq \emptyset$, iff $a \cup b \vdash_{DD} \perp$;
- argument d defends against argument $a = T \cup \Delta$, iff
 1. $d = T \cup \{\neg A\}$ ($d = T \cup \{A\}$) for some $A \in \Delta$ (respectively $\neg A \in \Delta$), or
 2. $d = T \cup \emptyset$ and $a \vdash_{DD} \perp$.

(Non-)Acceptability is defined as the least fixed point of an operator: given the set of binary relations \mathcal{R} over all sets of arguments in \mathcal{L} ,

- the *acceptability operator* $\mathcal{A}_T: \mathcal{R} \rightarrow \mathcal{R}$ is defined as follows: for any $acc \in \mathcal{R}$ and arguments a, a_0 : $(a, a_0) \in \mathcal{A}_T(acc)$ iff
 - $a \subseteq a_0$, or
 - for any argument b such that b attacks a ,
 - $b \not\subseteq a_0 \cup a$, and
 - there is argument d that defends against b such that $(d, a_0 \cup a) \in acc$.
- the *non-acceptability operator* $\mathcal{N}_T: \mathcal{R} \rightarrow \mathcal{R}$ is defined as follows: for any $nacc \in \mathcal{R}$ and arguments a, a_0 : $(a, a_0) \in \mathcal{N}_T(nacc)$ iff
 - $a \not\subseteq a_0$, and

- there is argument b such that b attacks a and
 - $b \subseteq a_0 \cup a$, or
 - for any argument d that defends against b , $(d, a_0 \cup a) \in nacc$.

ACC^T and $NACC^T$ denote the least fixed points of \mathcal{A}_T and \mathcal{N}_T respectively. We say that a is *acceptable wrt a_0 in T* iff $ACC^T(a, a_0)$, and a is *not acceptable wrt a_0 in T* iff $NACC^T(a, a_0)$.

The definition of entailment in AL is given as follows:

- a formula A in \mathcal{L} is *AL-entailed (from T , given \vdash_{DD})*, written $\models_{AL}^{DD(T)} A$, iff $ACC^T(\{A\}, \emptyset)$ and $NACC^T(\{\neg A\}, \emptyset)$.

In the remainder of the paper we often omit T from arguments and say that a formula A is acceptable (non-acceptable, respectively) when $ACC^T(\{A\}, \{\})$ holds ($NACC^T(\{A\}, \{\})$ holds, respectively).

4 An algebraic view of AL-entailment

Here and in the remainder of this section we assume \vdash_{DD} and (a directly consistent theory) T in \mathcal{L} as given, and that A, B are formulae in \mathcal{L} . (Note that T may be classically inconsistent).

We define $\chi^T : \mathcal{L} \rightarrow \mathcal{Z}_2$ corresponding to AL-entailment, as follows:

$$\chi^T(A) = 1 \text{ iff } \models_{AL}^{DD(T)} A$$

The following basic property of χ^T follows directly from the property of AL-entailment, that $T \vdash_{DD} A$ implies $ACC^T(\{A\}, \emptyset)$ and $NACC^T(\{\neg A\}, \emptyset)$:

Lemma 1. *If $T \vdash_{DD} A$ then $\chi^T(A) = 1$.*

The map χ^T satisfies axiomatic properties of ϕ depending on the inference rules in the underlying \vdash_{DD} . This correspondence follows from the following *closure property* of AL (see appendix B for a sketch of the proof):

Lemma 2. *if $\models_{AL}^{DD(T)} A$ and $T \cup \{A\} \vdash_{DD} B$, then $\models_{AL}^{DD(T)} B$.*

This result essentially indicates that AL respects its underlying basic inference rules (\vdash_{DD}). As a consequence, depending on the choice of \vdash_{DD} , χ^T satisfies various corresponding algebraic properties as considered in AnhomL. For example:

- if \vdash_{DD} includes the $\wedge E$ rule then
if $\chi^T(A \wedge B) = 1$ then $\chi^T(A) = 1$ and $\chi^T(B) = 1$.
- if \vdash_{DD} includes the $\wedge I$ rule then
if $\chi^T(A) = 1$ and $\chi^T(B) = 1$ then $\chi^T(A \wedge B) = 1$.

Hence when \vdash_{DD} includes both of the standard inference rules for conjunction then AL satisfies the multiplicative property:

Proposition 1. $\chi^T(A \wedge B) = \chi^T(A)\chi^T(B)$.

With respect to negation the properties of χ^T are:

- if $\chi^T(A) = 1$ then $\chi^T(\neg A) = 0$.

This follows since non-acceptability of $\neg A$ makes it impossible for $\neg A$ to be AL-entailed. Note though that the reverse property, i.e. if $\chi^T(A) = 0$ then $\chi^T(\neg A) = 1$, does not hold. This is similar to the properties of ϕ in the MS Scheme for AnhomL.

Similarly, if we include the $\vee I$ rule in \vdash_{DD} then the algebraic semantics of AL has the property:

- if $\chi^T(A)=1$ then $\chi^T(A \vee B)=1$.

In AnhomL the similar condition on ϕ holds in the MS Scheme due to the Boolean algebra structure of \mathcal{U} : we have $A \wedge (A \vee B) = A$ and so $\phi(A)\phi(A \vee B) = \phi(A)$, and thus if $\phi(A) = 1$ then $\phi(A \vee B) = 1$.

4.1 χ^T is anhomomorphic

Note that when $\neg(A \wedge B)$ holds in AL, i.e. $\chi^T(\neg(A \wedge B))=1$, then $\chi^T(A)=0$ or $\chi^T(B)=0$ (as they cannot both take the value 1). Does this impose that only one of the disjuncts will take the value 0 and hence the other will take the value 1 (i.e. that one of A or B will hold)? In other words, as in the AnhomL approach, is the map χ^T homomorphic in the full algebra of \mathcal{L}_2 , i.e. satisfying the additive property: $\chi^T(A \oplus B) = \chi^T(A) + \chi^T(B)$?

For disjoint formulae A, B , i.e. such that $\chi^T(\neg(A \wedge B))=1$, given, additionally, that $\chi^T(A \vee B)=1$ this homomorphic property (wrt + in \mathcal{L}_2) forces one of A and B to take the value 1: the disjunction can hold and yet neither disjunct holds.

When we take \vdash_{DD} to contain also the $\vee E$ inference rule (i.e. reasoning by cases) and the theory T is classically consistent then this property is satisfied (as AL and classical logic coincide in this case [4, 5, 6]) and hence χ^T is a homomorphism. However, in general, the answer to the questions above is no, as this property does not always hold in AL: the disjunction can hold and yet neither disjunct holds.

For the case of directly consistent theories that we are considering, if we exclude the $\vee E$ inference rule from \vdash_{DD} then a simple example shows how this property is violated:

Example 1. Let $T = \{\alpha \vee \beta, \alpha \rightarrow \perp, \beta \rightarrow \perp\}$. Then, $\chi^T(\alpha)=0$ and $\chi^T(\beta)=0$ although $\chi^T(\alpha \vee \beta)=1$.

Note that the well known ‘‘Barber of Seville’’ paradox is a concrete variant of this example:

$$T_{BS} = \{SBarber \vee SHimsel\!f, \neg(SBarber \wedge SHimsel\!f), SBarber \leftrightarrow SHimsel\!f\}.$$

The last formula in T_{BS} makes both $SBarber$ and $SHimsel\!f$ non-acceptable and hence both take the value 0 under χ_{BS}^T and yet $SBarber \vee SHimsel\!f$ and $SBarber \oplus SHimsel\!f$ take the value 1 under χ_{BS}^T as they are directly derivable from the theory.

The following example shows that even when $\vee E$ is present in \vdash_{DD} , χ^T is not homomorphic:

Example 2. $T = \{\alpha \vee \beta, \neg(\alpha \wedge \gamma), \neg(\alpha \wedge \neg\gamma), \neg(\beta \wedge \delta), \neg(\beta \wedge \neg\delta)\}$.

Again, both $\chi^T(\alpha)=0$ and $\chi^T(\beta)=0$ although $\chi^T(\alpha \vee \beta)=1$.

We see that in general, AL gives rise to an answering map on \mathcal{L} that shares properties comparable to the properties of the co-events in the MS Scheme for AnhomL. Through this, AL avoids classical logic paradoxes, that arguably do not exist in common sense reasoning. Since AnhomL was proposed as a logical framework for physics motivated by the need to encompass results in quantum mechanics that seem to many paradoxical by the lights of classical reasoning, it is interesting to investigate and compare how both AnhomL and AL treat an example from Quantum Physics.

5 An illustrative example from Quantum Physics

In the 3-slit experiment [8], particles (photons say) are incident on a barrier in which there are three equally spaced, parallel slits, labelled A, B and C . Beyond the barrier is a screen on which the particles are detected if they make it through the slits. The experiment is run and a particle is detected at a particular position P on the screen. The distance from the barrier to the screen and the slit spacing are such that the amplitude for the particle to pass through slit B and arrive at P is equal to the amplitude for the particle to pass through slit C and arrive at P , and minus the amplitude for the particle to pass through

slit A and arrive at P . There are three atomic propositions which are, “the particle passed through slit A ”, “the particle passed through slit B ” and “the particle passed through slit C ”. Treating this experiment in AnhomL, the event algebra is $\mathcal{U} = \{\emptyset, A, B, C, A \vee B, B \vee C, C \vee A, A \vee B \vee C\}$ where we use A to denote “the particle passed through slit A ” *etc.*

Quantum destructive interference – cancellation of equal and opposite amplitudes – means that the *quantum measure* [8] of each event $A \vee B$ and $A \vee C$ is zero and the *preclusion* condition [3, 8] implies that those events are denied by every allowed co-event: $\phi(A \vee B) = \phi(A \vee C) = 0$. Multiplicativity in AnhomL then implies also that $\phi(A) = \phi(B) = \phi(C) = 0$. The *minimality* condition [7] in the MS Scheme means that there is exactly one allowed co-event, ϕ , in this example: $\phi(B \vee C) = \phi(A \vee B \vee C) = 1$ are the only affirmations, all other events are denied by ϕ .

In AL, we represent this experiment and the quantum dynamics by $T = \{A \vee B \vee C, \neg(A \vee B), \neg(A \vee C)\}$ and \vdash_{DD} given by $\wedge I$, $\wedge E$ and $\vee I$. Here, T is a directly consistent theory (since $\vee E$ is not included in \vdash_{DD}).

Then A, B and C on their own are non-acceptable and hence $\chi^T(A) = \chi^T(B) = \chi^T(C) = 0$. Yet $\chi^T(A \vee B \vee C) = 1$.

We can also see that $\neg A$ is acceptable as there are no attacks against this apart from $T \cup \{A\}$, defended by $T \cup \{\neg A\}$ trivially, and apart from attacks that contain the negation of some direct consequence, D , of $\neg A$. But then the attack can be defended by $T \cup \{D\}$ which is acceptable wrt $\{\neg A\}$. Hence $\chi^T(\neg A) = 1$. Similarly, $\chi^T(\neg B) = 1$ and $\chi^T(\neg C) = 1$ hold.

Regarding $B \vee C$ this can be shown to be acceptable for the same reason as above, i.e. that there are no attacks against this except those that contain the negation of a direct consequence of $B \vee C$. The same holds for $\neg(B \vee C)$. Therefore χ^T gives 0 for both. Here we see a difference with the MS Scheme in AnhomL in which $\phi(B \vee C) = 1$.

6 Conclusions

We have initiated a comparison of two attempts to address the limitations of classical logic, one in the realm of commonsense reasoning and logical paradoxes and the other in quantum physics. In Argumentation Logic the central notion of acceptability of a formula gives a logical framework where the so called “tetralemma” [9] is naturally accommodated. In the future we will seek to understand better the relationship between AL and AnhomL by studying further quantum examples such as the Kochen-Specker theorem in AnhomL [2]. Quantum mechanics is sometimes described as “counter-intuitive” and “paradoxical”: it would be striking if understanding it requires an approach to logic that is actually closer to human, commonsense reasoning than the rigid rules of classical logic.

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A Natural Deduction

We use (some of) the following inference rules, for any propositional formulae A, B, C in \mathcal{L} :

$$\begin{array}{l} \wedge I : \frac{A, B}{A \wedge B} \quad \wedge E : \frac{A \wedge B}{A} \quad \wedge E : \frac{A \wedge B}{B} \quad \vee I : \frac{A}{A \vee B} \quad \vee I : \frac{B}{A \vee B} \quad \rightarrow I : \frac{[A \dots B]}{A \rightarrow B} \quad \perp I : \frac{A, \neg A}{\perp} \\ \neg E : \frac{\neg \neg A}{A} \quad \neg I/RA : \frac{[A \dots \perp]}{\neg A} \quad \vee E : \frac{A \vee B, [A \dots C], [B \dots C]}{C} \quad \rightarrow E : \frac{A, A \rightarrow B}{B} \end{array}$$

where $[\zeta, \dots]$ is a (sub-)derivation with ζ referred to as the *hypothesis*. $\neg I$ is also called Reduction ad Absurdum (RA). \perp stands for inconsistency.

B Sketch of proof of lemma 2

We need to show that (i) $ACC^T(\{B\}, \emptyset)$ and (ii) $NACC^T(\{\neg B\}, \emptyset)$.

(i) Any attack, $T \cup \{C\}$, against $\{B\}$ is also an attack against $\{A\}$ since $T \cup \{A\} \vdash_{DD} B$. Then the defence against C given from $ACC^T(\{A\}, \emptyset)$ will also form a defence for the acceptability of $\{B\}$. Otherwise, there will be an attack A' against some defence D' in the acceptability tree of A such that $A' \subseteq \text{Branch}(D') \cup \{B\}$ where $\text{Branch}(D')$ is the union of defences up to and including D' in the acceptability tree of A . But then $A'' = (A' - \{B\}) \cup \{A\}$ will also be an attack against D' (since $T \cup \{A\} \vdash_{DD} B$) such that $A'' \subseteq \text{Branch}(D')$, thus contradicting the acceptability of A .

(ii) The set $\{A\}$ is an attack against $\{\neg B\}$ (since $T, A \vdash_{DD} B$). Then because $ACC^T(\{A\}, \emptyset)$ holds it follows that $T \cup A \not\vdash_{DD} \perp$ and hence the only possible defence against A is $\{\neg A\}$. From $NACC^T(\{\neg A\}, \emptyset)$ then (when $B \neq A$ as is our case here) $NACC^T(\{\neg A\}, \{\neg B\})$ would also hold. Otherwise, if it does not hold then in the non-acceptability proof of $\{\neg event\}$ there will be some defence equal to $\{\neg B\}$ and so $NACC^T(\{\neg B\}, \{\neg A\})$ will hold. But then from this we have that $NACC^T(\{\neg B\}, \emptyset)$ also holds, as required, since if $NACC^T(\{\neg B\}, \{\neg A\})$ comes from an attack containing $\neg A$ then (when this is not part of the branch) its possible defence $\{A\}$ is attacked by $\{\neg B\}$ (since $T \cup \{A\} \vdash_{DD} B$) and therefore this defence will also be non-acceptable in the non-acceptability proof of $\{\neg B\}$ wrt the empty set. \square

SQEMA with Universal Modality

Dimitar Georgiev

Sofia University “St. Kliment Ohridski”
E-mail: dimitertg@yahoo.com

Abstract

SQEMA is an algorithm which, for a given formula of the basic modal language, in many cases is able to find its first-order correspondent. In this paper we augment SQEMA to the basic modal language extended with the universal modality.

1 Introduction

The problem with the existence of first-order correspondent formulas for modal formulas was proposed by van Benthem. This problem is not computable, as shown by Chagrova in her PhD thesis in 1989, see [2]. However, there have been solutions for some modal formulas. The most famous class of formulas for which there is a first-order correspondent is the Sahlqvist class, where one can use the Sahlqvist-van Benthem algorithm as described in [1] to obtain first-order correspondents.

There have been other algorithms to find first-order correspondents, for example, in [6] Gabbay and Ohlbach introduced the SCAN algorithm, and in [10], Szalas introduced DLS. SCAN is based on a resolution procedure applied on a Skolemized translation of the modal formula into the first-order logic, while DLS works on the same translation, but is based on a transformation procedure using a lemma by Ackermann. Both algorithms use a procedure of unskolemization, which is not always successful.

In [4] and [5] another algorithm for computing first-order correspondents in modal logic was introduced, called SQEMA, which is based on a modal version of the Ackermann Lemma. The algorithm works directly on the modal formulas without translating them into a first-order logic and without using Skolemization. SQEMA succeeds not only on all Sahlqvist formulas, but also on the extended class of *inductive formulas* introduced in [9]. There are examples of modal formulas on which SQEMA succeeds, while both SCAN and DLS fail, e.g.: $(\Box(\Box p \leftrightarrow q) \rightarrow p)$.

As proved in [4] and [5], SQEMA only succeeds on d-persistent (and hence, by, e.g. [1], canonical) formulas, i.e., whenever successful, it not only computes a local first-order correspondent of the input modal formula, but also proves its canonicity and therefore the canonical completeness of the modal logic axiomatized with that formula. This accordingly extends to any set of modal formulas on which SQEMA succeeds. Thus, SQEMA can also be used as an automated prover of canonical model completeness of modal logics.

An implementation of SQEMA in Java was given in [7]. Some additional simplifications were added to the implementation thanks to a suggestion by Renate Schmidt, which helps the implementation succeed on formulas such as $((\Box\Diamond p \rightarrow \Diamond\Box p) \vee (\Box p \rightarrow \Diamond p))$.

In this paper, we augment the SQEMA algorithm to $ML(\Box, [U])$, the basic modal language extended by adding the universal modality. We have a proof that formulas on which the new SQEMA+U succeeds are strongly complete and that the resulting first-order formula is frame-correspondent of the input formula.

2 Preliminaries

In Section 1 we discussed [4] where the SQEMA algorithm is described in detail and its properties are proved. This article is an extension of the previous work in [4] and will rely on the definitions and proofs given there. We now introduce SQEMA for the language $ML(\Box, [U])$, or the basic modal language extended by adding the universal modality, and we will refer to this variant of SQEMA as $SQEMA+U$.

We take a fixed countable set of *propositional variables* PROP. Formulas of $ML(\Box, [U])$ are defined inductively as $\phi = p | \top | \perp | \neg\psi_1 | (\psi_1 \vee \psi_2) | (\psi_1 \wedge \psi_2) | (\psi_1 \rightarrow \psi_2) | \Diamond\psi_1 | \Box\psi_1 | \langle U \rangle \psi_1 | [U]\psi_1$, where $p \in \text{PROP}$, ψ_1 and ψ_2 are already defined formulas. A Kripke frame is a tuple $\mathcal{F} = \langle W, R \rangle$, where W is a non-empty set and $R \subseteq W \times W$ is the *accessibility relation*. A *valuation* V is a mapping from the propositional variables to subsets of W , and a *model* \mathcal{M} is a tuple $\langle \mathcal{F}, V \rangle$ of a Kripke frame and a valuation on it. The *extension* of ϕ in \mathcal{M} , $\llbracket \phi \rrbracket_{\mathcal{M}}$, is: $\llbracket p \rrbracket_{\mathcal{M}} = V(p)$, $\llbracket \top \rrbracket_{\mathcal{M}} = W$, $\llbracket \perp \rrbracket_{\mathcal{M}} = \emptyset$, $\llbracket \neg\psi_1 \rrbracket_{\mathcal{M}} = W \setminus \llbracket \psi_1 \rrbracket_{\mathcal{M}}$, $\llbracket (\psi_1 \vee \psi_2) \rrbracket_{\mathcal{M}} = \llbracket \psi_1 \rrbracket_{\mathcal{M}} \cup \llbracket \psi_2 \rrbracket_{\mathcal{M}}$, $\llbracket (\psi_1 \wedge \psi_2) \rrbracket_{\mathcal{M}} = \llbracket \psi_1 \rrbracket_{\mathcal{M}} \cap \llbracket \psi_2 \rrbracket_{\mathcal{M}}$, $\llbracket (\psi_1 \rightarrow \psi_2) \rrbracket_{\mathcal{M}} = (W \setminus \llbracket \psi_1 \rrbracket_{\mathcal{M}}) \cup \llbracket \psi_2 \rrbracket_{\mathcal{M}}$, $\llbracket \Box\psi_1 \rrbracket_{\mathcal{M}} = \{w \in W \mid (\forall y \in W)(wRy \Rightarrow y \in \llbracket \psi_1 \rrbracket_{\mathcal{M}})\}$, $\llbracket \Diamond\psi_1 \rrbracket_{\mathcal{M}} = \{w \in W \mid (\exists y \in W)(wRy \ \& \ y \in \llbracket \psi_1 \rrbracket_{\mathcal{M}})\}$, $\llbracket [U]\psi_1 \rrbracket_{\mathcal{M}} = \{w \in W \mid (\forall y \in W)(y \in \llbracket \psi_1 \rrbracket_{\mathcal{M}})\}$, $\llbracket \langle U \rangle \psi_1 \rrbracket_{\mathcal{M}} = \{w \in W \mid (\exists y \in W)(y \in \llbracket \psi_1 \rrbracket_{\mathcal{M}})\}$. If $w \in \llbracket \phi \rrbracket_{\mathcal{M}}$ for some $w \in W$ and all models \mathcal{M} over a Kripke frame $\mathcal{F} = \langle W, R \rangle$, then ϕ is *valid* in \mathcal{F} , w ; $\mathcal{F}, w \Vdash \phi$. Two formulas are *locally equivalent* ($\phi \equiv \psi$), iff, in any model, their extensions are equal to one another.

As in [4], we extend $ML(\Box, [U])$ to $ML^+(\Box, [U])$, adding *nominals* and *reversed modalities* (\Box^{-1}, \Diamond^{-1}). Nominals are a new countable set of symbols, evaluated with singletons. Reversed modalities use R^{-1} instead of R for a given Kripke frame.

A *pure formula* is a formula which does not contain propositional variables but may contain nominals. A formula, which does not contain implications and where negations only occur immediately before propositional variables, is a formula in *negation normal form*. A formula, which is an implication whose right-hand side and left-hand side are both in negation normal form, is called an *equation* due to an allusion to the Gaussian elimination method. *Syntactically closed* is a formula where all occurrences of nominals and \Diamond^{-1} are positive, and all occurrences of \Box^{-1} are negative. *Syntactically open* is a formula where all occurrences of \Diamond^{-1} and nominals are negative, and all occurrences of \Box^{-1} are positive.

We consider a first-order language with formal equality, a binary predicate symbol R , and no other predicate symbols, no functional symbols and no constants. We denote this language as L_0 and we call L_0 formulas *FOL* formulas. A Kripke frame $\mathcal{F} = \langle W, R \rangle$ is also a structure for L_0 , interpreting the symbol R with the relation R . We inductively define the *standard translation* (ST) for pure formulas into first-order formulas of L_0 , where the resulting FOL formula will always have $n + 1$ free variables, where n is the number of distinct nominals in the pure formula. In this ST, we always allocate a countably infinite subset $YVAR = \{y_1, \dots\}$ of the FOL individual variables VAR and designate them to each of the nominals of $ML^+(\Box, [U])$, so our translation is unambiguous. A modal formula ϕ and an L_0 formula ψ with at most one free variable x are *locally correspondent* iff for every Kripke frame $\mathcal{F} = \langle W, R \rangle$, and for every $w \in W$: $\mathcal{F}, w \Vdash \phi$ iff $\mathcal{F} \models \psi_x[w]$.

3 The Algorithm SQEMA+U

Following [4], we define SQEMA+U in the following way:

INPUT: $\phi \in ML(\Box, [U])$

OUTPUT: $\langle \text{success}, \neg \text{pure}, \text{fol}(\phi) \rangle$ or $\langle \text{failure} \rangle$

STEP 1: Negate ϕ and rewrite $\neg\phi$ in negation normal form. Then, distribute the diamonds \diamond and $\langle U \rangle$ and conjunctions over disjunctions as much as possible, using the local equivalences:

Rule 1.1: $\diamond_0(\gamma_1 \vee \gamma_2) \equiv (\diamond_0\gamma_1 \vee \diamond_0\gamma_2)$ for \diamond_0 among \diamond and $\langle U \rangle$

Rule 1.2: $(\gamma_1 \vee \gamma_2) \wedge \gamma_3 \equiv (\gamma_1 \wedge \gamma_3) \vee (\gamma_2 \wedge \gamma_3)$

Thus, obtain $\neg\phi \equiv \bigvee_k \psi_k$ where no further applications of rules 1.1 or 1.2 are possible on any ψ_k . The algorithm now reserves the nominal $i_0 \in \text{NOM}$, it does not occur in any ψ_k (which is trivially true because in our case, $\psi_k \in \text{ML}(\Box, [U])$) and it will be used throughout the steps. The algorithm now proceeds with STEP 2, applied separately on each of the subformulas ψ_k , and if it succeeds for all ψ_k , it will proceed to STEP 6. Otherwise, if even one of the branches for a single k fails, the algorithm returns $\langle failure \rangle$ as output and stops.

STEP 2: Let ψ be one of the disjuncts from STEP 1. Now, consider the equation $i_0 \rightarrow \psi$, where $i_0 \in \text{NOM}$ is the reserved in STEP 1 nominal. Start solving a system of equations that only contains $(i_0 \rightarrow \psi)$ by proceeding to STEP 3.

STEP 3: From the current system, eliminate every propositional variable that occurs only positively or negatively throughout the system, by replacing it with \top or \perp , respectively.

STEP 4: Non-deterministically pick an elimination order for the remaining propositional variables in the current system. Try eliminating each variable in order, by proceeding to STEP 5. If any elimination order succeeds, and thus, all propositional variables have been eliminated from the current system, proceed to STEP 6. If all elimination orders fail, report failure for the current system and return to STEP 1.

STEP 5: Take the propositional variable p that has to be eliminated as input from STEP 4. Apply rules for converting the current system, as to eliminate any occurrences of p . The rules to use will be listed below. If p has been eliminated, report success and return the current system to STEP 4 to try eliminating the remaining variables. If elimination fails, backtrack, as will be described in detail below, and try STEP 5 again. If all attempts fail, report failure to eliminate p and resume executing STEP 4.

STEP 6: If this step is reached by all branches of the execution, then all propositional variables have been eliminated from all systems resulting from the negation of the input formula. In each system, take the conjunction of all equations, and then take the disjunctions of all pure formulas obtained from all systems, to finally obtain a pure formula *pure*. Then, form the formula $(\forall y_1 \dots \forall y_n \exists x_0) \text{ST}(\neg \text{pure}, x_0)$, where y_1, \dots, y_n are all occurring variables corresponding to the nominals in *pure*, except for y_0 , corresponding to the designated current state nominal i_0 . y_0 is left free because we are computing the local FOL correspondent. Return the result $\langle success, \neg \text{pure}, (\forall y_1 \dots \forall y_n \exists x_0) \text{ST}(\neg \text{pure}, x_0) \rangle$.

Now, we will discuss the rules that the algorithm will use in STEP 5 above.

Transformation Rules: This is the detailed description of STEP 5's rules. Denote the propositional variable that needs to be eliminated with p . Apply rules for converting the current system, so that in the end we get a current system with only three classes of formulas: 1. alphas, 2. betas, and 3. thetas for p . Alphas will be of the form $\alpha_j \rightarrow p$ for $1 \leq j \leq n$, betas will be of the form $\beta_k(p)$ for $1 \leq k \leq m$, and thetas will be of the form θ_l for $1 \leq l \leq q$, such that:

1. p does not occur in any α_j part of any alpha;
2. all betas are negative in p ;
3. there are no occurrences of p in any theta.

If the current system is successfully split into three types of formulas as described above, then the current system is replaced with a new current system $\{\beta_1(p/(\alpha_1 \vee \dots \vee \alpha_n)), \dots, \beta_m(p/(\alpha_1 \vee \dots \vee \alpha_n)), \theta_1, \dots, \theta_q\}$, thus p is eliminated. Note that, if n or m were 0, then rule 8, described below, or STEP 3 would have eliminated p , so we assume non-zero n and m . We call this elimination rule the Ackermann rule. If one of the newly obtained formulas above, ψ ,

is not in equation form, rewrite it to negation normal form as ψ' , then replace it with $\top \rightarrow \psi'$.

The rules that the algorithm can use to obtain a split system as described above, are:

1. \wedge -rule. Replace $\beta \rightarrow (\gamma \wedge \delta)$ with $\beta \rightarrow \gamma, \beta \rightarrow \delta$.
2. Left-shift \vee -rule. Replace $\beta \rightarrow (\gamma \vee \delta)$ with $(\beta \wedge \neg \gamma) \rightarrow \delta$. This includes a non-deterministic choice between γ and δ , and backtracking if the first choice fails.
3. Right-shift \vee -rule. Replace $(\beta \wedge \neg \gamma) \rightarrow \delta$ with $\beta \rightarrow (\gamma \vee \delta)$.
4. Left-shift \Box -rule. For any $\Box_0 \in \{\Box, [U], \Box^{-1}\}$, replace $\gamma \rightarrow \Box_0 \delta$ with $\Diamond_0^{-1} \gamma \rightarrow \delta$, where \Diamond_0^{-1} is the corresponding to \Box_0 reversed diamond.
5. Right-shift \Box -rule. For any $\Diamond_0^{-1} \in \{\Diamond^{-1}, \langle U \rangle, \Diamond\}$, replace $\Diamond_0^{-1} \gamma \rightarrow \delta$ with $\gamma \rightarrow \Box_0 \delta$, where \Box_0 is the corresponding to \Diamond_0^{-1} un-reversed box.
6. \Diamond -rule. For any $\Diamond_0 \in \{\Diamond, \langle U \rangle, \Diamond^{-1}\}$, replace $j \rightarrow \Diamond_0 \gamma$ with $j \rightarrow \Diamond_0 k, k \rightarrow \gamma$, where k is a new nominal that hasn't been used anywhere in any system in any branch of this run of SQEMA+U.

7. Reasoning rules. Freely replace any equation with a locally equivalent equation, if:

- 7.1. The result is an equation and is locally equivalent to the replaced one.
- 7.2. If before the replacement, the left-hand side of the equation was syntactically closed and the right-hand side was syntactically open, the same holds for the replacement.

Example rules of type 7:

- Commutativity and associativity of \wedge and \vee
- Convert a formula into negation normal form
- Replace $\gamma \vee \neg \gamma$ with \top , and $\gamma \wedge \neg \gamma$ with \perp
- Replace $\gamma \vee \top$ with \top , and $\gamma \vee \perp$ with γ
- Replace $\gamma \wedge \top$ with γ , and $\gamma \wedge \perp$ with \perp
- Replace $\gamma \rightarrow \perp$ with $\neg \gamma$ and $\gamma \rightarrow \top$ with \top
- Replace $\perp \rightarrow \gamma$ with \top and $\top \rightarrow \gamma$ with γ and vice versa, when it is needed to obtain an equation after a variable elimination

- Replace $\neg \Diamond_0 \neg \gamma$ with $\Box_0 \gamma$ and $\neg \Box_0 \neg \gamma$ with $\Diamond_0 \gamma$ for corresponding modalities.

Rules that affect $\langle U \rangle$ and $[U]$ are listed in the table below. For $j \in \{1, 2\}$, we abbreviate U_j for either $[U]$ or $\langle U \rangle$, we abbreviate \Diamond for either \vee or \wedge , we abbreviate \Box^ε for either \Box or \Box^{-1} , and we abbreviate \Diamond^ε for either \Diamond or \Diamond^{-1} .

Replace	with	Replace	with
$(i_1 \rightarrow \langle U \rangle i_2), i_1, i_2 \in \text{NOM}$	\top	$\langle U \rangle i$, where $i \in \text{NOM}$	\top
$U_1 U_2 \gamma$	$U_2 \gamma$	$\langle U \rangle \gamma_1 \vee \gamma_2$, for $\gamma_2 \equiv \neg \gamma_1$	\top
$\Box^\varepsilon U_1 \gamma$	$U_1 \gamma \vee \Box^\varepsilon \perp$	$\langle U \rangle \gamma \vee \Diamond^\varepsilon \gamma$	$\langle U \rangle \gamma$
$[U](U_1 \gamma_1 \Diamond U_2 \gamma_2)$	$U_1 \gamma_1 \Diamond U_2 \gamma_2$	$[U] \gamma \vee \Box^\varepsilon \gamma$	$\Box^\varepsilon \gamma$
$[U](U_1 \gamma_1 \Diamond \gamma_2)$	$U_1 \gamma_1 \Diamond [U] \gamma_2$	$[U] \gamma_1 \wedge \gamma_2$, for $\gamma_2 \equiv \neg \gamma_1$	\perp
$[U] \neg i$, where $i \in \text{NOM}$	\perp	$\langle U \rangle \gamma \wedge \Diamond^\varepsilon \gamma$	$\Diamond^\varepsilon \gamma$
$\Diamond^\varepsilon U_1 \gamma$	$U_1 \gamma \wedge \Diamond^\varepsilon \top$	$[U] \gamma \wedge \Box^\varepsilon \gamma$	$[U] \gamma$
$\langle U \rangle (U_1 \gamma_1 \wedge U_2 \gamma_2)$	$U_1 \gamma_1 \wedge U_2 \gamma_2$	$[U] \gamma \wedge \gamma$	$[U] \gamma$
$\langle U \rangle (U_1 \gamma_1 \wedge \gamma_2)$	$U_1 \gamma_1 \wedge \langle U \rangle \gamma_2$	$\langle U \rangle \gamma \wedge \gamma$	γ

8. Positive and negative variables rule: As in STEP 3, replace any variable that occurs only positively in the system with \top , and replace any variable that occurs only negatively in the system with \perp .

If STEP 5 fails to eliminate p , backtrack, try the other choice for any Left-shift \vee -rule that is still pending. If this fails, backtrack, change the polarity of p by substituting $\neg p$ for p in the starting system, convert all formulas to equations as described above, and try STEP 5 again. If

this also fails, report failure to eliminate p and resume STEP 4. If p has been eliminated, report success and return the current system to STEP 4 to try eliminating the remaining variables.

Theorem 1. *Let ϕ be an $\text{ML}(\Box, [U])$ formula and let $\text{SQEMA}+U$ return $(\text{success}, \neg\text{pure}, \text{fol}(\phi))$ on ϕ . Then, the following hold: (1) ϕ and $\text{fol}(\phi)$ are locally correspondent; (2) The normal modal logic $K_U + \phi$ is strongly complete with respect to the class of frames validating $\text{fol}(\phi)$.*

Proof. We use a special case of descriptive general frames and d-persistence for $\text{ML}(\Box, [U])$ similarly to [1]. The proof of (1) and that ϕ is d-persistent is analogous to the proof given in [4]. The axioms and completeness of K_U are given in [8]. Similarly to [1], with some additional work because the canonical relation for $[U]$ is an S_5 relation, we prove that there is a descriptive general canonical sub-frame for $K_U + \phi$ and any MCS, thus proving (2).

4 Implementation Notes

The implementation of $\text{SQEMA}+U$ is based on the existing servlet-based SQEMA implementation from 2006. It is written in Java. The program runs deterministically and always terminates. $\text{SQEMA}+U$ is compiled with the Google GWT compiler to Javascript, runs directly in the browser, and displays the result if $\text{SQEMA}+U$ succeeds. The user can see the proof of the result if the run was successful, and can also see the full log of the actions taken by the program. The website URL is <http://www.fmi.uni-sofia.bg/fmi/logic/sqema>. $\text{SQEMA}+U$ is vNext, while SQEMA is 0.9.2.2. A complete version of the proofs for $\text{SQEMA}+U$ for $\text{ML}(\Box, [U])$ can be obtained from the author in PDF format.

4.1 Future Work

- Prove that $\text{SQEMA}+U$ succeeds on all $\text{ML}(\Box, [U])$ Sahlqvist formulas, similarly to [4].
- Extend $\text{SQEMA}+U$ to $\text{ML}^+(\Box, [U])$ inputs (with proofs).
- Implement the polyadic version of SQEMA .

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Total Equivalence Systems for Classes of 3-valued Projection Logic whose Projections Equal to the Class of Linear Boolean Functions

Olga Gerasimova and Ilya Makarov

Department of Data Analysis and Artificial Intelligence
National Research University Higher School of Economics
E-mail: olga.g3993@gmail.com, iamakarov@hse.ru

Abstract

We give a complete description of total equivalence systems (TES) for formulas based on closed classes of functions from the projection logic $P_{3,2}$ with the property that the restrictions of its functions to the set $\{0, 1\}$ constitute a closed class of linear Boolean functions. For each such class, we find a total equivalence system, providing an algorithm for a transformation of an arbitrary formula to its canonical form.

1 Introduction

1.1 Closed classes and generating sets

We consider a subset $P_{3,2}$ of 3-valued logic functions of several variables defined on the set $\{0, 1, 2\}$ and taking values in the set $\{0, 1\}$ or being a single variable. For a system of functions $\mathfrak{A} \in P_{3,2}$ we denote by $[\mathfrak{A}]$ the closure of \mathfrak{A} relative to superposition and insertion of nonessential variable operations for formulas (terms) based on \mathfrak{A} . If $[\mathfrak{A}] = \mathfrak{A}$ holds, then \mathfrak{A} is said to be a *closed class*. A subset $\alpha \subset P_{3,2}$ is called a *generating system* if $[\alpha] = \mathfrak{A}$ holds. If we consider a closed class of $P_{3,2}$, and allow all of its functions possess only values from $\{0, 1\}$, then we would obtain a *projection* of a closed class of 3-valued logic, which itself is a closed class relative to Boolean logic.

D. Lau in [2] gave a complete list and presented examples of generating systems for every closed class of $P_{3,2}$ whose projection is the class L of linear Boolean functions. There is a countable set of such classes. Denoting the inclusion-maximal class by $\text{pr}^{-1}L$, we enumerate such classes and their generating systems:

$$\text{pr}^{-1}L = [A \cup \{j_1(x) + j_1(x_1)j_1(x_2)j_2(x_3), j_1(x) + j_2(y)\}], \quad (1)$$

$$A \in Z_{2,0} \cap \text{pr}^{-1}L, [\text{pr} A] = L$$

$$L_2 = [1, j_1(x) + j_1(y), j_1(x) \cdot j_2(y)] \quad (2)$$

$$L_{2,r} = \begin{cases} [1, j_1(x) + j_1(y), j_{2,r}(x_1, \dots, x_r) = j_2(x_1) \cdot \dots \cdot j_2(x_r)], r < \infty \\ [\{1, j_1(x) + j_1(y)\} \cup \bigcup_{q \geq 1} j_2(x_1) \cdot \dots \cdot j_2(x_q)], r = \infty \end{cases} \quad (3)$$

$$Z_{2,i} \cap \text{pr}^{-1}L = [j_i(x) + j_i(y), 1], i \in \{0, 1\} \quad (4)$$

1.2 Total equivalence systems

Two formulas $\Phi(x_1, \dots, x_n)$ and $\Psi(y_1, \dots, y_m)$ are called *equivalent* if they realize equal functions. *Equivalence on α* is a formal equality $\Phi = \Psi$ of *equivalent* formulas on α . By $\Sigma(\alpha)$ we

denote the *set of all equivalences* on α . The *closure* Σ of $\Sigma_0 \subseteq \Sigma(\alpha)$ is defined by the reflexivity, symmetry, transitivity rules (the same as for the equivalence relation); the substitution rule, which allows substituting an arbitrary formula for every occurrence of some variable in both parts of the equivalence from Σ ; the deduction rule, which constructs a new equivalence by taking formula Φ , φ being its subformula, and using an already obtained equivalence “ $\varphi = \psi$ ” $\in \Sigma$ to obtain a new equivalence $\Phi = \Psi$, where Ψ is obtained by substituting ψ for a fixed occurrence of φ in Φ .

A system of equivalences Σ is said to be a *total equivalence system* (TES) for a closed class \mathfrak{A} generated by α if $[\Sigma] = \Sigma(\alpha)$. So a closed class \mathfrak{A} generated by subset α has a finite TES (an FTES) Σ_0 if and only if, for any two equivalent formulas Φ and Ψ on α , there exists a finite deduction sequence of pairwise equivalent formulas Θ_i , $i = 1, \dots, n$, such that the system $[\Sigma_0]$ contains the following equivalences: $\Phi = \Theta_1, \Theta_1 = \Theta_2, \dots, \Theta_{n-1} = \Theta_n, \Theta_n = \Psi$.

2 Main results

R. C. Lyndon in [1] proved that the existence of TES for a given closed class \mathfrak{A} does not depend on the chosen generating system of \mathfrak{A} (for details, see [3]).

In this paper we prove the existence of finite total equivalence systems for $\text{pr}^{-1}L, L_2, L_{2,r}$ and non-existence of an FTES for $L_{2,\infty}$, which has only a countable TES. The second author has previously proved in [3] the existence of FTES for (4) in a general case.

We use the separator method mentioned in [3] to obtain canonical forms for a formula considering corresponding projection part of Boolean cube with tuples of ‘0’, ‘1’ and “remainder” part that contains at least one value ‘2’ in a tuple. For every class we describe TES and an algorithm for converting an arbitrary formula into its canonical form.

For all $a \in \{0, 1, 2\}$ we define the functions $j_a(z) \in P_{3,2}$ as the Boolean indicator of condition under which “ $z = a$ ”.

2.1 Class $\text{pr}^{-1}L$

We know from [2] that an arbitrary function G from the maximal subclass $\text{pr}^{-1}L$ of $P_{3,2}$ whose projection is the class L can be represented by the following formula

$$G^{CF} = \alpha_0 + \sum_{i=1}^n \alpha_i \cdot j_1(x_i) + P_0(x_1, \dots, x_n) + \sum_I \sum_J k(x_{t_{|J|+1}}, \dots, x_{t_n}) \cdot P_{I,J}(x_{t_1}, \dots, x_{t_{|J|}}),$$

$$P_{I,J}(x_{t_1}, \dots, x_{t_{|J|}}) = \sum_{k=1}^{|J|} \sum_{t_1 < \dots < t_k} \beta_{k,t_1, \dots, t_k} \cdot j_2(x_{t_1}) \cdot \dots \cdot j_2(x_{t_k}), \quad \alpha_i, \beta_{m, \dots} \in \{0, 1\},$$

$$k(x_{t_{|J|+1}}, \dots, x_{t_n}) = j_1(x_{t_{|J|+1}}) \cdot \dots \cdot j_1(x_{t_n}), \quad P_0(x_1, \dots, x_n) = P_{\emptyset, \{1, \dots, n\}}(x_1, \dots, x_n),$$

where $\emptyset \neq I \subseteq \{1, \dots, n\}$ is a tuple of indices for variables being arguments of j_1 and $J = \{1, \dots, n\} \setminus I$.

Statement 1. *For any formula from $\text{pr}^{-1}L$ there is a unique canonical formula of the form written above.*

Proof of Statement 1. Suppose that an arbitrary function F from $\text{pr}^{-1}L$ has two different canonical forms F_1^{CF} and F_2^{CF} . Let us compute their sum them by modulo 2:

$$0 = F + F = F_1^{CF} + F_2^{CF} = G$$

For each tuple γ , $G(\gamma) = 0$ should hold. Let us consider the term in formula G in which the number of occurrences of j_2 is minimal (possibly, zero) and among all such conjunctions let

us choose the conjunction with the lowest (possibly, zero) number of j_1 . Calculating G on a tuple with values 2 assigned to the variables in j_2 functions, values 1 assigned to the variables in j_1 functions, and all other variables taking value 0, we get a contradiction: $G = 1$ on some tuple. Therefore, all coefficients of the considered terms must be equal to 0, and $G \equiv 0$. Thus we prove that formulas F_1^{CF} and F_2^{CF} are graphically equal with lexicographic ordering of conjunctions in sum, and F from $\text{pr}^{-1}L$ has a unique canonical form. ■

The problem is that the conjunction is not a linear function and can not be used to construct the canonical form for our class.

Using a description of a generating system for (1), we introduce a new function $j_{112}(x, y, z) = j_1(x) \cdot j_1(y) \cdot j_2(z) \in \text{pr}^{-1}L$ and obtain the following new generating set and canonical form for $\text{pr}^{-1}L$:

$$\alpha = [0, 1, x + y = j_1(x) \oplus j_1(y), j_1(x), j_{112}(x, y, z)]$$

$$F^{CF} = \beta_0 + \sum_{i=1}^n \beta_i \cdot j_1(x_i) + \sum_I \sum_J \beta_{I,J} \cdot F_{k,n-k}(x)$$

$$K_{I,J}(x) = \bigwedge_{i=1}^k j_1(x_i) \cdot \bigwedge_{j=1}^{n-k} j_2(y_j) \Leftrightarrow F_{k,n-k} = j_{112}(F_{k-1,n-k-1}, x_k, y_{n-k})$$

$$\underbrace{F_{0,n-2k} = j_{112}(F_{0,n-2k-1}, x_1, y_{n-2k})}_{\text{repeating } x_1, \text{ when } k \leq n/2}; \underbrace{F_{n-2k,0} = j_{112}(F_{n-2k-1}, x_{n-2k}, y_1)}_{\text{repeating } y_1, \text{ when } k > n/2}; F_{0,0} = 1.$$

Polynomial P_0 is obtained as polynomial of j_{112} by substituting 1 for the first two arguments. All other possible combinations of I and J can be obtained by repetition of variables for the second and the third arguments. Thus it is a one-to-one mapping between G^{CF} and F^{CF} .

Now we present the equivalences that will form an FTES for $\text{pr}^{-1}L$ and describe an algorithm for reducing formulas to the canonical type (table 1).

#	Description	Equivalences
1	Replace variables by j_1	(1) $x + y = j_1(x) + j_1(y)$; (2) $x + 0 = j_1(x)$;
2	Eliminate substitutions of the function $x+y$	(3) $j_{112}(x_1 + x_2, y, z) = j_{112}(x_1, y, z) + j_{112}(x_2, y, z)$; (4) $j_{112}(x, y, z) = j_{112}(y, x, z)$;
3	Omit the function j_1 where it is possible	(4); (5)-(9) $j_1(f_i) = f_i$, for all $f_i \in \alpha$; (10) $j_{112}(j_1(x), y, z) = j_{112}(x, y, z)$;
4	Rewrite to the left-handed formula	(4); (11) $j_{112}(j_{112}(x_1, x_2, y_1), j_{112}(x_3, x_4, y_2), y_3) = j_{112}(j_{112}(j_{112}(x_1, x_2, y_1), x_3, y_2), x_4, y_3)$;
5	Order variables lexicographically	(12) $j_{112}(j_{112}(x, y, z), w, t) = j_{112}(j_{112}(x, w, z), y, t)$; (13) $j_{112}(j_{112}(x, y, z), w, t) = j_{112}(j_{112}(x, y, t), w, z)$;
6	Eliminate repeating variables in formulas where it is possible	(4); (12); (14) $j_{112}(x, x, z) = j_{112}(1, x, z)$; (15) $j_{112}(j_{112}(x, y, z), y, z) = j_{112}(x, y, z)$; (16) $j_{112}(j_{112}(x, y, z), y, t) = j_{112}(j_{112}(x, y, z), x, t)$;
7	Order a sum	(17) $x + y = y + x$; (18) $(x + y) + z = x + (y + z)$;
8	Apply reducing equivalences	(19)-(23) $j_{112}(x, y, f_i) = 0$, for all $f_i \in \alpha$; (24)-(26) $j_{112}(0, x, y) = 0$; $j_{112}(x, y, x) = 0$; $x + x = 0$.

 Table 1: FTES for $\text{pr}^{-1}L$

2.2 Class L_2

Using generating system (2) properties for associativity and commutativity of such defined addition we present a description of all formulas realizing function of L_2 :

$$G^{CF} = \sum_{i=1}^n \alpha_i \cdot j_1(x_i) \cdot P_i(j_2(x_1), \dots, \widehat{j_2(x_i)}, \dots, j_2(x_n)) + \alpha_{n+1} \cdot P_{n+1}(j_2(x_1), \dots, j_2(x_n)),$$

where $P_i(j_2(x_1), \dots, \widehat{j_2(x_i)}, \dots, j_2(x_n)) = \alpha_0 + \sum_{k=1}^{n-1} \sum_{\substack{t_1 < \dots < t_k \\ \forall j, t_j \neq i}} \alpha_{k, t_1, \dots, t_k} \cdot j_2(x_{t_1}) \cdots j_2(x_{t_k})$.

We can see that there are at most $n+1$ Zhegalkin's polynomials placed in conjunction with formulas $j_1(x_i)$ or constant in outer sum. Each Zhegalkin's polynomial depends on functions $j_2(x_j)$ instead of variables $x_j, j \neq i$ for $i \neq n+1$:

Such construction heavily uses the conjunction, the projection of which is not a linear function. We introduce a new function $j_{1 \wedge 2}(x, y) = j_1(x) \cdot j_2(y)$, which belongs to L_2 , suppose that $x_{n+1} \equiv 1$, and construct the canonical form for L_2 in a different generating system

$$\alpha = [0, 1, j_1(x), j_2(x), j_{1 \wedge 2}(x, y), x + y = j_1(x) \oplus j_1(y)]$$

$$F^{CF} = \sum_{i=1}^n \alpha_i \cdot j_1(x_i) + \sum_J \alpha_{i,J} \cdot K_{i,J}(x)$$

$$H_{i,k} \Leftrightarrow K_{i,J}(x) = j_1(x_i) \cdot \bigwedge_{l=1, l \neq i}^k j_2(x_{t_l})$$

$$H_{i,k} = j_{1 \wedge 2}(H_{i,k-1}, x_k); \quad H(i, 0) = j_1(x_i), \quad \{t_1, \dots, t_k\} = J \subseteq \{1, \dots, n\} \setminus \{i\}$$

Now we show the equivalences that will form an FTES for L_2 and describe an algorithm for reducing formulas to the canonical type (table 2).

#	Description	Equivalences
1	Replace variables by j_1	(1) $x + y = j_1(x) + j_1(y)$; (2) $x + 0 = j_1(x)$;
2	Eliminate substitutions of the function $x+y$	(3) $j_1(x + y) = j_1(x) + j_1(y)$; (4) $j_{1 \wedge 2}(x + y, z) = j_{1 \wedge 2}(x, z) + j_{1 \wedge 2}(y, z)$;
3	Omit the function j_1 where it is possible	(5)-(10) $j_1(f_i) = f_i$, for all $f_i \in \alpha$; (11) $j_{1 \wedge 2}(j_1(x), y) = j_{1 \wedge 2}(x, y)$;
4	Order variables lexicographically	(12) $j_{1 \wedge 2}(j_2(x), y) = j_{1 \wedge 2}(j_2(y), x)$; (13) $j_{1 \wedge 2}(j_{1 \wedge 2}(x, y), z) = j_{1 \wedge 2}(j_{1 \wedge 2}(x, z), y)$;
5	Eliminate repeating variables	(14) $j_{1 \wedge 2}(j_{1 \wedge 2}(x, y), y) = j_{1 \wedge 2}(x, y)$;
6	Order a sum	(15) $x + y = y + x$; (16) $(x + y) + z = x + (y + z)$;
7	Apply reducing equivalences	(17)-(22) $j_{1 \wedge 2}(x, f_i) = 0$, for all $f_i \in \alpha$; (23)-(25) $x + x = 0$; $j_{1 \wedge 2}(x, x) = 0$; $j_{1 \wedge 2}(0, x) = 0$.

Table 2: FTES for L_2

2.3 Class $L_{2,r}, r < \infty$

Considering generating system (3) we obtain a canonical form for $L_{2,r}$ as a sum of linear parts of j_1 functions and Zhegalkin's polynomial of degree not greater than r depending on j_2 functions. We have no conjunction in our class, so we introduce new functions $j_{2,q}(x_1, \dots, x_q) = j_2(x_1) \cdots j_2(x_q), q \in 1, \dots, r$, and write a unique canonical form in a new generating set:

$$\beta = [0, 1, x + y = j_1(x) \oplus j_1(y), j_1(x), j_{2,q}(x_1, \dots, x_q), \text{ for } q \in \overline{1, \dots, r}]$$

$$F^{CF}(x_1, x_2, \dots, x_n) = \beta_0 + \sum_{i=1}^n \beta_i \cdot j_1(x_i) + \sum_{k=1}^r \sum_{t_1 < \dots < t_k} \beta_{k, t_1, \dots, t_k} \cdot j_{2,k}(x_{t_1}, \dots, x_{t_k})$$

We define equivalences, which will be contained in FTES for $L_{2,r}$ and describe an algorithm for a transformation of an arbitrary formula to its canonical form (table 3).

#	Description	Equivalences
1	Replace variables by j_1	(1) $x + y = j_1(x) + j_1(y)$; (2) $x + 0 = j_1(x)$;
2	Omit the function j_1	(3)-(6+r) $j_1(f_i) = f_i$, for all $f_i \in \alpha$;
3	Order variables lexicographically	(7+r)-(5+2r) $j_{2,q}(x_1, x_2, \dots, x_{q-1}, x_q) = j_{2,q}(x_1, x_2, \dots, x_q, x_{q-1}) = \dots = j_{2,q}(x_2, x_1, \dots, x_q)$;
4	Eliminate repeating variables where	(6+2r)-(4+3r) $j_{2,q}(x_1, x_2, \dots, x_{q-1}, x_{q-1}) = j_{2,q-1}(x_1, x_2, \dots, x_{q-1})$, $q = 1, \dots, r$
5	Order a sum	(5+3r) $x + y = y + x$; (6+3r) $(x + y) + z = x + (y + z)$;
6	Apply reducing equivalences	(7+3r)-(10+4r) $j_{2,q}(f_i, x_2, \dots, x_q) = 0$, for all $f_i \in \alpha$; (11+4r) $x + x = 0$.

Table 3: FTES for $L_{2,r}$

2.4 Class $L_{2,r}, r = \infty$

We formulate a property C_r as follows: a formula F satisfies the property C_r if in the functions $j_{2,q}$ used in the formula construction q is from the set $\{1, \dots, r\}$. For each formula from $L_{2,\infty} \setminus (Z_{2,1} \cap \text{pr}^{-1} L)$ there exists minimal r such that F satisfies C_r .

Statement 2. *For the formulas from $L_{2,r}, r = \infty$ the property C_r is preserved under application of superposition and insertion of nonessential variable operations.*

Proof of Statement 2. Let us take two arbitrary formulas $A(x, \bar{y})$ and B from $L_{2,\infty}$, both with the property C_r . There are two cases for the formula $D = A(B, \bar{y})$: it is either 0 (substituting for $j_{2,q}$) or $A + B$ (substituting for sum). It is clear that property C_r is preserved. The insertion of nonessential variables obviously preserves C_r . ■

Theorem 1. *The class $L_{2,\infty}$ has no FTES, but has only a countable TES.*

Proof of Theorem 1. At first, let us take an arbitrary finite system of equivalences Σ . There exists a minimal fixed r equal to the degree of the formula such that both parts of each equivalence from Σ satisfy the C_r property. Then the equivalence $j_{2,r+1}(x_1, \dots, x_r, x_{r+1}) = j_{2,r+1}(x_1, \dots, x_{r+1}, x_r)$ does not belong to $[\Sigma]$ obtained from Σ by superpositions and inclusions of nonessential variables for left or right parts of equivalences.

Secondly, in order to obtain a countable TES we have to take the countable union of finite equivalence systems for $L_{2,r}, r = 1, 2, \dots$. Obviously, it is a countable set, and taking an arbitrary formula F we get minimal r such that F realizes a function from $L_{2,r}$. ■

3 Conclusions

We consider all closed classes of $P_{3,2}$ whose projection is the class L of linear Boolean functions. For all such classes we find TES and construct canonical forms. We prove the existence of finite TES for all classes, excluding one class, which has countable TES.

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Extended Contact Algebras and Internal Connectedness

Tatyana Ivanova

Sofia University, Sofia, Bulgaria
E-mail: tatyana.m.ivanova@gmail.com

Abstract

We prove a representation theorem of contact algebras with added predicate “internal connectedness”.

1 Introduction

This paper is related to region-based theory of space (RBTS). The roots of this theory can be found in some philosophical ideas of Whitehead[23] and de Laguna[6]. Recent researches are [1],[3],[18],[22], researches concerning various applications are [4], [5]. One of the most popular system related to Qualitative Spatial Reasoning is the system of Region Connection Calculus (RCC) introduced in [19]. An algebraic reformulation of RCC as a Boolean algebra with an additional relation C - contact, was presented in [20]. The simplest contact algebra was introduced in [7]. Representation theory for contact algebras corresponding to RCC was given for the first time in [9], representation theory for contact algebras corresponding to various important classes of topological spaces, was given in [7]. Contact algebras have also non-topological models based on the notion of adjacency space (see [10], [8], [2]). In [21] is presented a complete quantifier-free axiomatization of several logics on region-based theory of space, based on contact relation and connectedness predicates c and $c^{\leq n}$, and completeness theorems for the logics in question are proved. It was shown in [21] that c and $c^{\leq n}$ are definable in contact algebras by the contact C . The predicates c and $c^{\leq n}$ were studied for the first time in [16], [17] (see also [22]). The expressiveness and complexity of spatial logics containing c and $c^{\leq n}$ has been investigated in [12], [15], [14], [11], [13]. In the present paper we consider the predicate c° - internal connectedness, and prove that this predicate cannot be defined in the language of contact algebras. Because of this we add to the language a new ternary predicate symbol \vdash which has the following sense: in the contact algebra of regular closed sets of some topological space $a, b \vdash c$ iff $a \cap b \subseteq c$. It turns out that the predicates c° , C can be defined in the new language. We prove a representation theorem of Boolean algebras with added relations \vdash , C and c° .

2 Internal connectedness

Definition 1. Following [7], by a contact algebra we mean any system $\underline{B} = (B, C) = (B, 0, 1, \cdot, +, *, C)$, where $(B, 0, 1, \cdot, +, *)$ is a nondegenerate Boolean algebra, $*$ denotes the complement, and C is a binary relation in B , called a contact, such that

$$(C1) aCb \rightarrow a, b \neq 0$$

$$(C2) aC(b + c) \leftrightarrow aCb \text{ or } aCc$$

$$(C3) aCb \rightarrow bCa$$

$$(C4) a.b \neq 0 \rightarrow aCb$$

An example of contact algebra is a contact algebra of regular closed sets. Let X be a topological space. A subset a of X is regular closed if $a = Cl(Int(a))$. The set of all regular

closed subsets of X is denoted by $RC(X)$. It is known that the regular closed sets with $a \leq b \leftrightarrow a \subseteq b$, $a+b = a \cup b$, $a.b = Cl(Int(a \cap b))$, $a^* = Cl(-a)$, $0 = \emptyset$, $1 = X$, $aCb \leftrightarrow a \cap b \neq \emptyset$ form a contact algebra.

Let $c^o(x)$ means that $Int(x)$ is a connected topological space in the subspace topology.

Proposition 1. *There does not exist a formula $A(x)$ in the language of contact algebras such that: for every regular closed subset x of some topological space, $c^o(x)$ iff $A(x)$ is valid in the algebra of regular closed subsets of the topological space.*

Proof: Suppose that there exists a formula $A(x)$ in the language of contact algebras such that: for every regular closed subset x of some topological space, $c^o(x)$ iff $A(x)$ is valid in the algebra of regular closed subsets of the topological space.

We consider the topological space (X, O) , where $X = \{1, 2, 3, 4, 5, 6, 7\}$ and the topology is defined by an open basis: $\{\{1, 2, 3\}, \{7\}, \{3, 6, 7\}, \{2, 5, 7\}, \{2\}, \{3\}, X, \emptyset\}$. It can be easily verified that the regular closed sets are $\{4, 5, 6, 7\}, \{1, 2, 3, 4, 5, 6\}, \{1, 2, 4, 5\}, \{1, 3, 4, 6\}, \{1, 3, 4, 5, 6, 7\}, \{1, 2, 4, 5, 6, 7\}, X, \emptyset$.

We consider the subspace of X , $Y = X \setminus \{1\}$. The regular closed subsets of Y are: $\{4, 5, 6, 7\}, \{2, 3, 4, 5, 6\}, \{2, 4, 5\}, \{3, 4, 6\}, \{3, 4, 5, 6, 7\}, \{2, 4, 5, 6, 7\}, Y, \emptyset$.

We define function f from $RC(X)$ to $RC(Y)$ in the following way:

$$f(t) = \begin{cases} t & \text{if } 1 \notin t \\ t \setminus \{1\} & \text{if } 1 \in t \end{cases}$$

f is an isomorphism from $(RC(X), \leq, \emptyset, X, \cdot, +, *, C)$ to $(RC(Y), \leq, \emptyset, Y, \cdot, +, *, C)$.

Let $a = \{1, 2, 3, 4, 5, 6\}$. a is internally connected.

Let $b = \{2, 3, 4, 5, 6\}$. $Int_Y b = \{2, 3\}$. We will prove that b is not internally connected. $\{2, 3\}$ is not connected, since $\{2, 3\} = \{2\} \cup \{3\}$, $\{2\}$ and $\{3\}$ are closed in $\{2, 3\}$. Consequently b is not internally connected.

We have $a \in RC(X)$, $c^o(a)$. Consequently $A(a)$. Now consider the topological space Y . Using $b \in RC(Y)$ and $\neg c^o(b)$, we have $\neg A(b)$. We also have $b = f(a)$. $(RC(X), \leq, \emptyset, X, \cdot, +, *, C)$ and $(RC(Y), \leq, \emptyset, Y, \cdot, +, *, C)$ are isomorphic structures for the language of contact algebras, A is a formula in the same language. Consequently $A(a)$ iff $A(f(a))$ - a contradiction. \square

Let X be a topological space. We define the relation \vdash in $RC(X)$ in the following way $a, b \vdash c$ iff $a \cap b \subseteq c$. For every $a, b \in RC(X)$, $aCb \leftrightarrow a, b \not\vdash 0$.

Proposition 2. *Let X be a topological space. For every a in $RC(X)$, $c^o(a)$ iff $\forall b \forall c (b \neq 0 \wedge c \neq 0 \wedge a = b + c \rightarrow b, c \not\vdash a^*)$.*

Proof: Only the (\leftarrow) part is interesting. Suppose that $\forall b \forall c (b \neq 0 \wedge c \neq 0 \wedge a = b + c \rightarrow b, c \not\vdash a^*)$. Suppose that $\neg c(Int_X a)$. Consequently there are b_1, c_1 - closed in $Int_X a$, such that $Int_X a = b_1 + c_1$, $b_1 \neq \emptyset$, $c_1 \neq \emptyset$, $b_1 \cap c_1 = \emptyset$. $b_1 = b \cap Int_X a$, $c_1 = c \cap Int_X a$, where b and c are closed in X . Let $b' = Cl_X b_1$, $c' = Cl_X c_1$. We have $a = b' \cup c'$. $Int_X b' = b_1$ and hence b' is regular closed. Similarly c' is regular closed. $b', c' \neq \emptyset$; $a = b' + c'$; so $b' \cap c' \not\subseteq a^* = -Int_X a$; so $b_1 \cap c_1 \neq \emptyset$ - a contradiction. Consequently $c(Int_X a)$. \square

3 Extended contact algebras

Definition 2. *Extended contact algebra (ECA, for short) is a system $\underline{B} = (B, \leq, 0, 1, \cdot, +, *, \vdash, C, c^o)$, where $(B, \leq, 0, 1, \cdot, +, *)$ is a nondegenerate Boolean algebra, \vdash is a ternary relation in*

B such that the following axioms are true:

- (1) $a, b \vdash c \rightarrow b, a \vdash c$,
- (2) $a \leq b \rightarrow a, a \vdash b$,
- (3) $a, b \vdash a$,
- (4) $a, b \vdash x, a, b \vdash y, x, y \vdash c \rightarrow a, b \vdash c$,
- (5) $a, b \vdash c \rightarrow a.b \leq c$,
- (6) $a, b \vdash c \rightarrow a + x, b \vdash c + x$,

C is a binary relation in B , defined by $aCb \leftrightarrow a, b \neq 0$. c° is a unary predicate in B , defined by $c^\circ(a) \leftrightarrow \forall b \forall c (b \neq 0 \wedge c \neq 0 \wedge a = b + c \rightarrow b, c \neq a^*)$.

If $\underline{B} = (B, \leq, 0, 1, \cdot, +, *, \vdash, C, c^\circ)$ is an ECA, then C satisfies the axioms of contact $C1 - C4$.

Let X be a topological space. Axioms (1) – (6) are true in $RC(X)$.

Definition 3. Let $(B, \leq, 0, 1, \cdot, +, *, \vdash, C, c^\circ)$ be an ECA and $S \subseteq B$.

$$S \vDash_0 x \stackrel{def}{\leftrightarrow} x \in S$$

$$S \vDash_{n+1} x \stackrel{def}{\leftrightarrow} \exists x_1, x_2 : x_1, x_2 \vdash x, S \vDash_{k_1} x_1, S \vDash_{k_2} x_2, \text{ where } k_1, k_2 \leq n$$

$$S \vDash x \stackrel{def}{\leftrightarrow} \exists n : S \vDash_n x$$

Lemma 1. If $\{x_1\} \cup S \vDash y, \{x_2\} \cup S \vDash y$, then $\{x_1 + x_2\} \cup S \vDash y$.

Proof: It can be proved that $\{x_1\} \cup S \vDash y$ implies $\{x_1 + x_2\} \cup S \vDash y + x_2$ by axiom (6). Similarly from $\{x_2\} \cup S \vDash y$ we get $\{x_2 + y\} \cup S \vDash y + y$, i.e. $\{y + x_2\} \cup S \vDash y$. From here and $\{x_1 + x_2\} \cup S \vDash y + x_2$ we obtain $\{x_1 + x_2\} \cup S \vDash y$. \square

Lemma 2. Let $S = \{a_1, a_2, \dots, a_n\} \cup \{b_1, b_2, \dots, b_k\}$ for some $n, k > 0$ and $S \vDash x$. Let $a = a_1.a_2 \dots .a_n, b = b_1.b_2 \dots .b_k$. Then $a, b \vdash x$.

Proof: By induction on n we will prove that $\forall n \forall x (S \vDash_n x \rightarrow a, b \vdash x)$

Case 1: $n = 0$

Suppose that $S \vDash_0 x$. Consequently $x \in S$. Without loss of generality $x = a_1$. $a, a \vdash a_1, a, b \vdash a$; so $a, b \vdash a_1$ by axiom (4).

Case 2: $n > 0$

Suppose that $S \vDash_n x$. Using the unduction hypothesis, we prove that $a, b \vdash x$. \square

Definition 4. Let $\underline{B} = (B, \leq, 0, 1, \cdot, +, *, \vdash, C, c^\circ)$ be an ECA. A subset of B Γ is an abstract point if the following conditions are satisfied:

- 1) $1 \in \Gamma$
- 2) $0 \notin \Gamma$
- 3) $a + b \in \Gamma \rightarrow a \in \Gamma$ or $b \in \Gamma$
- 4) $a, b \in \Gamma, a, b \vdash c \rightarrow c \in \Gamma$

Note that ultrafilters are abstract points.

Lemma 3. Let $\underline{B} = (B, \leq, 0, 1, \cdot, +, *, \vdash, C, c^\circ)$ be an ECA. Let $A \neq \emptyset, A \subseteq B, a \in B, A \neq a$. Then the set (M, \subseteq) , where $M = \{P \subseteq B : A \subseteq P; a \notin P; x, y \in P, x, y \vdash z \rightarrow z \in P\}$, has a maximal element Γ and Γ is an abstract point.

Proof: Let $P_0 = \{t : A \vDash t\}$. $P_0 \in M$. Every chain in (M, \subseteq) has an upper bound in M . By the Zorn lemma, (M, \subseteq) has a maximal element Γ . Let a_1 be an arbitrary element of A . $a_1 \leq 1$; so $a_1, a_1 \vdash 1$; so $1 \in \Gamma$. From $0, 0 \vdash a$ and $a \notin \Gamma$ it follows that $0 \notin \Gamma$. Suppose that $x + y \in \Gamma$. Suppose that $\{x\} \cup \Gamma \vDash a$ and $\{y\} \cup \Gamma \vDash a$. By lemma 1, $\{x + y\} \cup \Gamma \vDash a$, i.e. $\Gamma \vDash a$; so $a \in \Gamma$ - a contradiction. Consequently $\{x\} \cup \Gamma \not\vDash a$ or $\{y\} \cup \Gamma \not\vDash a$. Without loss of generality $\{x\} \cup \Gamma \not\vDash a$. Let $\Gamma' = \{z : \{x\} \cup \Gamma \vDash z\}$. We have $\Gamma \subseteq \Gamma', \Gamma' \in M, \Gamma$ is a maximal element of (M, \subseteq) ; so $\Gamma = \Gamma'$ and hence $x \in \Gamma$. Consequently Γ is an abstract point. \square

Theorem 1. (*Representation theorem*) Let $\underline{B} = (B, \leq, 0, 1, \cdot, +, *, \vdash, C, c^o)$ be an ECA. Then there is a compact, T_0 topological space X and an embedding h of \underline{B} into $RC(X)$.

Proof: Let X be the set of all abstract points of \underline{B} and for $a \in B$, suppose $h(a) = \{\Gamma \in X : a \in \Gamma\}$. The set $\{h(a) : a \in B\}$ can be taken as a closed basis for a topology of X . As in the proof of the representation theorem for Boolean contact algebras [22] we prove that $h(0) = \emptyset$, $h(1) = X$, $h(a + b) = h(a) + h(b)$ and that $h(a) \subseteq h(b)$ iff $a \leq b$. In a similar way as in [22] we prove that $h(a^*) = Cl(-h(a))$ for every $a \in B$, $h(a)$ is a regular closed set. $h(a \cdot b) = h((a^* + b^*)^*) = (h(a)^* + h(b)^*)^* = h(a) \cdot h(b)$. Obviously $a, b \vdash c \rightarrow h(a), h(b) \vdash h(c)$. Suppose that $h(a), h(b) \vdash h(c)$. Suppose that $a, b \not\vdash c$. Suppose that $\{a, b\} \vDash c$. By lemma 2, $a, b \vdash c$ - a contradiction. Consequently $\{a, b\} \not\vDash c$. By lemma 3, there is an abstract point Γ such that $a, b \in \Gamma$, $c \notin \Gamma$ - a contradiction with $h(a), h(b) \vdash h(c)$. Consequently $a, b \vdash c$. Obviously $aCb \leftrightarrow h(a)Ch(b)$. Clearly $c^o(h(a))$ implies $c^o(a)$. Let $c^o(a)$. Suppose that $\neg c^o(h(a))$. Consequently there are b, c such that $b, c \neq \emptyset$, $h(a) = b \cup c$, $b \cap c \subseteq h(a)^* = h(a^*)$. b and c are closed, so $b = \bigcap_{i \in I} h(b_i)$, $c = \bigcap_{j \in J} h(c_j)$ for some sets I, J . Let $A_1 = \{b_i : i \in I\} \cup \{c_j : j \in J\}$. Suppose that $A_1 \vDash a^*$. There is a finite nonempty subset A_2 of A_1 , such that $A_2 \vDash a^*$. Let $b_{i_1} \in \{b_i : i \in I\}$, $c_{j_1} \in \{c_j : j \in J\}$. Let $A_2 = \{b_{i_2}, b_{i_3}, \dots, b_{i_k}\} \cup \{c_{j_2}, c_{j_3}, \dots, c_{j_l}\}$ for some $k, l \geq 1$. Let $b' = b_{i_1} \cdot b_{i_2} \cdot \dots \cdot b_{i_k}$, $c' = c_{j_1} \cdot c_{j_2} \cdot \dots \cdot c_{j_l}$. $\{b_{i_1}, b_{i_2}, \dots, b_{i_k}\} \cup \{c_{j_1}, c_{j_2}, \dots, c_{j_l}\} \vDash a^*$. By lemma 2, $b', c' \vdash a^*$. Suppose that $b'.a = 0$. Consequently $h(b_{i_1}) \cdot \dots \cdot h(b_{i_k}) \cdot h(a) = \emptyset$; so $ClInt(h(b_{i_1}) \cap \dots \cap h(b_{i_k}) \cap h(a)) = \emptyset$; so $Int(h(b_{i_1}) \cap \dots \cap h(b_{i_k}) \cap h(a)) = \emptyset$; so $Int(b) = \emptyset$; so $b = ClInt(b) = \emptyset$ - a contradiction. Consequently $b'.a \neq 0$ (1). Similarly $c'.a \neq 0$ (2). $a \leq b_{i_m} + c_{j_n}$ for every $m = 1, \dots, k$, $n = 1, \dots, l$. Consequently $a \leq b' + c'$. We have $a = b'.a + c'.a$ (3). From $b', c' \vdash a^*$ we obtain $b'.a, c'.a \vdash a^*$. From $c^o(a)$, (1), (2), (3) we get $b'.a, c'.a \not\vdash a^*$ - a contradiction. Consequently $A_1 \not\vDash a^*$. By lemma 3, there is an abstract point Γ_1 such that $A_1 \subseteq \Gamma_1$, $a^* \notin \Gamma_1$. $\Gamma_1 \in b$, $\Gamma_1 \in c$, $\Gamma_1 \notin h(a^*)$ - a contradiction with $b \cap c \subseteq h(a^*)$. Consequently $c^o(h(a))$. Consequently h is an embedding. As in the proof of the representation theorem for Boolean contact algebras [22] we prove that X is a compact, T_0 space. \square

4 Thanks

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Lukasiewicz Public Announcement Logic

Umberto Rivieccio

Delft University of Technology, Delft, The Netherlands
E-mail: `u.rivieccio@tudelft.nl`

This contribution reports on ongoing joint work with Ricardo Rodríguez (University of Buenos Aires, Argentina) and Leonardo Cabrer (University of Florence, Italy). We develop a logic of public announcements in a many-valued setting: for this logic we provide an algebraic and a Kripke-style semantics, proving completeness with respect to both. We also show how our approach can be extended to other systems of dynamic epistemic logic.

Dynamic logics are language expansions of (classical) modal logic designed to reason about changes induced by actions of different kinds, e.g. updates on the memory state of a computer, displacements of a moving robot, belief-revisions changing the common ground among different cognitive agents, knowledge update. Semantically, an action is represented by a transformation of a model describing a given state of affairs: this transformation shows how the state of affairs is modified after the action has been performed.

The logic of public announcements [13, 1, 4, 2] is a dynamic logic that models the epistemic change on the cognitive state of a group of agents resulting from a given fact (expressed by some proposition, say α) becoming publicly known. To each proposition α one associates a *dynamic* modal operator $\langle \alpha \rangle$, the semantic interpretation thereof is given by the transformation of models corresponding to its action-parameter α . Thus, $\langle \alpha \rangle \varphi$ is a proposition in the language of public announcement logic which is interpreted as saying that, *after α has been publicly announced*, the proposition φ holds.

Our work combines the logic of public announcements of [1] with the family of finite-valued Lukasiewicz modal logics studied in [8]. The recent papers [11, 10, 14, 15] introduce dynamic epistemic expansions of several logics weaker than classical logic, providing a semantic definition to introduce the extended logic. The main methodological novelty of these works is a dual characterization of epistemic updates via Stone-type dualities: here we adopt the same approach, further extending it from a mathematical point of view.

Epistemic updates induced by public announcements are formalized in relational (Kripke-style) models by means of the relativization construction, which creates a submodel of the original model. The corresponding submodel injection map is dually represented in [11] as a quotient construction between the complex algebras of the original model and of the updated one. This construction allows us to study epistemic updates within mathematical environments having a support that is more general than classical logic, as long as we can exploit an existing duality between the algebraic and the Kripke-style semantics of the logic.

Here we further generalize [11, 10] taking as propositional base the well-known family of finite-valued Lukasiewicz logics, while epistemic (i.e. *static*) modalities are modeled within the framework of the modal Lukasiewicz logics introduced in [8]. In this way we can build propositions such as $\langle \alpha \rangle \Box_i \varphi$ expressing the fact that, after α has been publicly announced, the agent i knows that φ holds. Each static modal operator of many-valued Łukasiewicz modal logic \Box_i (\Diamond_i) is used to represent the knowledge (beliefs) of an agent i .

The need for such a framework, which combines a many-valued approach with modalities, has been forcefully argued for in a number of recent works [5, 6, 3, 9]. The main idea can be summarized by saying that, while modal operators have proven to be an invaluable tool in a variety of logical applications, classical logic turned out to be not suitable in many reasoning contexts, especially those involving partial or contradictory information [14, 15], constructive

reasoning [11] and vagueness. Especially this last issue, namely the problem of graded properties and vague predicates, can be best dealt with in a many-valued framework such as that of Lukasiewicz logic. In this contribution we lay the mathematical foundations for such an approach.

We generalize the pseudo-quotient construction of [11, 14] to the algebraic semantics of Lukasiewicz modal logic (*modal MV-algebras*), which allows us to define a natural interpretation of the language of public announcement logic on these algebraic structures. In this way we establish which interaction axioms between dynamic modalities and the other connectives of the logic are sound with respect to our intended semantics. The resulting calculus defines a many-valued version of public announcement logic, which we prove to be complete with respect to our algebra-based semantics analogously to the classical case. We also introduce an equivalent relational semantics based on many-valued Kripke frames, which is obtained from the algebraic semantics via a Stone-type duality.

From a technical point of view, the main difficulties, and hence the novelty, of our approach stem from two sources.

Firstly, the existing definitions of the pseudo-quotient (which models the epistemic update on algebraic models of the logic) as such do not work on modal MV-algebras. This led us to restrict our attention (at least for the time being) to algebras having a non-modal reduct that belongs to a finitely-generated variety of MV-algebras, which allowed us to exploit the fact that any such algebra is k -potent for some finite natural number k . Our construction also provides insight on the possibility of defining pseudo-quotients on more general algebras, such as residuated lattices endowed with modal operators (corresponding to the logics considered e.g. in [3]). Essentially, what one needs is to be able to provide a “simple” equational characterization (such as that of [11, Section 3.2]) of the logical filter generated by any given element in the algebra.

Secondly, the algebraic approach to dynamic epistemic logics relies on the existence of a well-developed duality for the algebras corresponding to the logic, which in all existing studies is a Stone- or Priestley-type duality. It is well known that MV-algebras are not easily understood from the point of view of Stone- or Priestley-type duality theory (see, e.g., [12, 7]). In this work, since we restrict ourselves to k -potent MV-algebras, we have at hand the natural duality for finitely generated varieties of MV-algebras developed in [16] and its extension to modal MV-algebras [8]. We expand their algebraic and Kripke-style semantics to account for dynamic modalities.

As potential directions for future research, we would like to mention the issue of extending our treatment to arbitrary modal MV-algebras, and also that of extending our algebraic account of updates in order to be able to introduce and axiomatize a Lukasiewicz version of the logic of epistemic action and knowledge considered in [10].

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Part Restrictions in an Expressive Description Logic with Transitive Roles and Role Hierarchies

Mitko Yanchev

Sofia University “St. Kliment Ohridski”, Bulgaria
E-mail: yanchev@fmi.uni-sofia.bg

Abstract

In this paper we consider an extension of well-studied Description Logic (DL) \mathcal{ALCQIH}_{R^+} with concept constructors called *part restrictions* which realize rational grading. Having the ability to make statements about a part of a set of successors, part restrictions enrich the expressive capabilities of DLs. We combine the tableaux technique used for \mathcal{ALCQIH}_{R^+} with a specific technique for dealing with part restrictions to prove that the reasoning in the new DL $\mathcal{ALCQPIH}_{R^+}$ is decidable.

1 Introduction

Description Logics (DLs) are widely used in knowledge-based systems. The representation in the language of transitive relations, in different possible ways [10], is important for dealing with complex objects. *Transitive roles* permit such objects to be described by referring to their components, or ingredients without specifying a particular level of decomposition. The expressive power can be strengthened by allowing additionally *role hierarchies*. DL \mathcal{ALCH}_{R^+} [5], with both transitive roles and role hierarchies, is suitable for implementation, as, though having the same EXPTIME-complete worst-case reasoning complexity as other DLs with comparable expressivity, it is shown to be more amenable to optimization [4].

Inverse roles enable the language to describe both the whole by means of its components and vice versa, for example `has_part` and `is_part_of`. This syntax extension is captured in DL \mathcal{ALCIH}_{R^+} [7]. As a next step, in [7] the language is enriched with the counting (or *grading*—a term coming from the modal counterparts of DLs [3]) *qualifying number restrictions* what results in DL \mathcal{ALCQIH}_{R^+} . It is given a sound and complete decision procedure for that logic.

We go further considering concept constructors, which we call *part restrictions*, capable of distinguishing a part of a set of successors. These constructors are analogues of the modal operators for *rational grading* [11], which generalize the majority operators [9]. They are $MrR.C$ and (the dual) $WrR.C$, where r is a rational number in $(0, 1)$, R is a role, and C is a concept. The intended meaning of $MrR.C$ is ‘More than r -part of R -successors (or R -neighbours, in the presence of inverse roles) of the current object possess the property C ’. Part restrictions essentially enrich the expressive capabilities of DLs. From the ‘object domain’ point of view they seem to be more ‘socially’ than ‘technically’ oriented, but in any case they give new strength to the language. The usual example of the stand-alone use of part restrictions is to express the notion of qualifying majority in a voting system: $M\frac{2}{3}\text{voted.Yes}$.

On the other hand, *presburger constraints* in the language of Presburger Modal Logic [2], a many-relational language with independent relations, capture both integer and rational grading, and have rich expressiveness. The rational grading operators are expressible by the presburger constraints. Nonetheless, the presence of separate rational grading constructors in DLs also proves beneficial. Together with the use of a technique specially designed for exploring the rational grading, they allow following a common way for obtaining decidability and complexity results as in less, so in more expressive languages with rational grading. In particular, reasoning

complexity results—polynomial, NP, and co-NP—concerning a range of description logics from the \mathcal{AL} -family with part restrictions added, are obtained [13], [12], [14], as well as PSPACE-results for modal and expressive description logics [12], [15].

Now we consider the DL $\mathcal{ALCQPIH}_{R^+}$, the extension of \mathcal{ALCQIH}_{R^+} with part restrictions. We use the tableaux technique to prove that the reasoning in the extended logic is decidable.

2 Syntax and Semantics of $\mathcal{ALCQPIH}_{R^+}$

The $\mathcal{ALCQPIH}_{R^+}$ -syntax and semantics differ from those of \mathcal{ALCQIH}_{R^+} only in the presence of part restrictions. To keep clarity, we give here the extended definitions in whole.

Definition 1. Let \mathbf{C}_o be a set of concept names, \mathbf{R}_o be a set of role names, some of which transitive, and \mathbf{Q} be a set of rational numbers in $(0, 1)$. We denote the set of transitive role names \mathbf{R}^+ , so that $\mathbf{R}^+ \subseteq \mathbf{R}_o$. Then we define the set of $\mathcal{ALCQPIH}_{R^+}$ -roles (we will refer to simply as ‘roles’) as $\mathbf{R} = \mathbf{R}_o \cup \{R^- \mid R \in \mathbf{R}_o\}$, where R^- is the inverse role of R .

As the inverse relation on roles is symmetric, to avoid considering roles such as R^{--} we define a function Inv which returns the inverse of a role. Formally, $\text{Inv}(R) = R^-$, if R is a role name, and $\text{Inv}(R^-) = R$.

A role inclusion axiom has the form $R \sqsubseteq S$, for two roles R and S , and the acyclic inclusion relation \sqsubseteq . For a set of role inclusion axioms \mathcal{R} , a role hierarchy is:

$$\mathcal{R}^+ := (\mathcal{R} \cup \{\text{Inv}(R) \sqsubseteq \text{Inv}(S) \mid R \sqsubseteq S \in \mathcal{R}\}, \sqsubseteq^+),$$

where \sqsubseteq^+ is the transitive-reflexive closure of \sqsubseteq over $\mathcal{R} \cup \{\text{Inv}(R) \sqsubseteq \text{Inv}(S) \mid R \sqsubseteq S \in \mathcal{R}\}$.

A role R is simple with respect to \mathcal{R}^+ iff $R \notin \mathbf{R}^+$ and, for any $S \sqsubseteq^+ R$, $S \notin \mathbf{R}^+$.

The set of $\mathcal{ALCQPIH}_{R^+}$ -concepts (we will refer to simply as ‘concepts’) is the smallest set such that: 1. every concept name is a concept; 2. if C and D are concepts, and R is a role, then $\neg C$, $C \sqcap D$, $C \sqcup D$, $\forall R.C$, and $\exists R.C$ are concepts; 3. if C is a concept, R is a simple role, $n \geq 0$, and $r \in \mathbf{Q}$, then $\geq nR.C$, $\leq nR.C$, $MrR.C$, and $WrR.C$ are concepts.

The constraint roles in qualifying number restrictions, as well as in part restrictions to be simple is used essentially in the proofs. From the other side, the presence in the language of role hierarchies together with number restrictions on transitive roles leads to undecidability [8].

An interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consists of a nonempty set $\Delta^{\mathcal{I}}$, called the domain of \mathcal{I} , and a function $\cdot^{\mathcal{I}}$ which maps every concept to a subset of $\Delta^{\mathcal{I}}$ and every role to a subset of $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. For the ‘old’ constructors the mapping is the usual (see, e.g. in [7]). For part restrictions, for any concept C , role R , and rational number $r \in \mathbf{Q}$ the properties that follow hold. $R^{\mathcal{I}}(x)$ denotes the set of $R^{\mathcal{I}}$ -neighbours of x , $R^{\mathcal{I}}(x, C)$ denotes the set of $R^{\mathcal{I}}$ -neighbours of x which are in $C^{\mathcal{I}}$, i.e. $\{y \mid \langle x, y \rangle \in R^{\mathcal{I}} \text{ and } y \in C^{\mathcal{I}}\}$, and $\#M$ denotes the cardinality of a set M .

$$\begin{aligned} (MrR.C)^{\mathcal{I}} &= \{x \in \Delta^{\mathcal{I}} \mid \#R^{\mathcal{I}}(x, C) > r \cdot \#R^{\mathcal{I}}(x)\} \\ (WrR.C)^{\mathcal{I}} &= \{x \in \Delta^{\mathcal{I}} \mid \#R^{\mathcal{I}}(x, \neg C) \leq r \cdot \#R^{\mathcal{I}}(x)\} \end{aligned}$$

Also, for any role S and any role $R \in \mathbf{R}^+$ we define:

$$\begin{aligned} \langle x, y \rangle \in S^{\mathcal{I}} &\text{ iff } \langle y, x \rangle \in (\text{Inv}(S))^{\mathcal{I}} \\ \text{if } \langle x, y \rangle \in R^{\mathcal{I}} \text{ and } \langle y, z \rangle \in R^{\mathcal{I}}, &\text{ then } \langle x, z \rangle \in R^{\mathcal{I}} \end{aligned}$$

An interpretation \mathcal{I} satisfies a role hierarchy \mathcal{R}^+ iff $R^{\mathcal{I}} \subseteq S^{\mathcal{I}}$ for any $R \sqsubseteq S \in \mathcal{R}^+$; we denote that by $\mathcal{I} \models \mathcal{R}^+$.

A concept C is *satisfiable* with respect to a role hierarchy \mathcal{R}^+ iff there exists an interpretation \mathcal{I} such that $\mathcal{I} \models \mathcal{R}^+$ and $C^{\mathcal{I}} \neq \emptyset$. Such an interpretation is called a *model* of C with respect to \mathcal{R}^+ . For $x \in C^{\mathcal{I}}$ we say that the object x *satisfies* C , also that x is an *instance* of C , while $x \in \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$ *refuses* C . A concept D *subsumes* a concept C with respect to \mathcal{R}^+ (denoted $C \sqsubseteq_{\mathcal{R}^+} D$) iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds for every interpretation \mathcal{I} such that $\mathcal{I} \models \mathcal{R}^+$.

Thus, for $x \in \Delta^{\mathcal{I}}$, x is in $(MrR.C)^{\mathcal{I}}$ iff strictly greater than r part of $R^{\mathcal{I}}$ -neighbours of x satisfies C , and x is in $(WrR.C)^{\mathcal{I}}$ iff no greater than r part of $R^{\mathcal{I}}$ -neighbours of x refuses C .

Checking the subsumption between concepts is the most general reasoning task in DLs. From the other side, $C \sqsubseteq D$ iff $C \sqcap \neg D$ is unsatisfiable. So, in the presence of negation of an arbitrary concept, checking the (un)satisfiability becomes as complex as checking the subsumption.

In what follows we consider concepts to be in the *negation normal form* (NNF), in which negation can appear only in front of the concept names. We denote the NNF of $\neg C$ by $\sim C$.

3 A Tableau for $\mathcal{ALCQPIH}_{R^+}$

Let us firstly note that all considerations and techniques from [7] concerning \mathcal{ALCQIH}_{R^+} are applicable to the extended DL with only mild changes. So, in what follows we present explicitly, due to the restriction of space, only what is new, relying on and referring to [7] for the rest.

We extend the definition of \mathcal{ALCQIH}_{R^+} -tableau to define a tableau for $\mathcal{ALCQPIH}_{R^+}$ by changing one property to reflect the presence of part restrictions, and adding two new ones, and we prove Lemma 1. T is a tableau for a concept D , \mathcal{L} is a function which maps each individual of T to a set of subconcepts of D , and \mathcal{E} is a function, mapping each role occurring in D to a set of pairs of individuals. The sets of neighbours of an individual s in T are denoted as in an interpretation. In property 13. (modified property 11. from the definition of \mathcal{ALCQIH}_{R^+} -tableau) \odot is a placeholder for $\geq n$, $\leq n$, Mr and Wr , for arbitrary $n \geq 0$ and $r \in \mathbf{Q}$.

11. if $MrR.C \in \mathcal{L}(s)$, then $\sharp R^T(s, C) > r \cdot \sharp R^T(s)$ (new),
12. if $WrR.C \in \mathcal{L}(s)$, then $\sharp R^T(s, \sim C) \leq r \cdot \sharp R^T(s)$ (new),
13. if $\odot R.C \in \mathcal{L}(s)$ and $\langle s, t \rangle \in \mathcal{E}(R)$, then either $C \in \mathcal{L}(t)$, or $\sim C \in \mathcal{L}(t)$.

Lemma 1. *An $\mathcal{ALCQPIH}_{R^+}$ -concept D is satisfiable with respect to a role hierarchy \mathcal{R}^+ iff there exists a tableau for D with respect to \mathcal{R}^+ .*

4 Constructing an $\mathcal{ALCQPIH}_{R^+}$ -Tableau

Lemma 1 guarantees that the algorithm which constructs tableaux for $\mathcal{ALCQPIH}_{R^+}$ -concepts can serve as a decision procedure for concept satisfiability (and hence, also for subsumption between concepts) with respect to a role hierarchy \mathcal{R}^+ . We present such an algorithm. It uses the blocking technique [6], namely *pair-wise blocking* as defined in [7], to ensure only finite although exponential paths in the completion tree. It also uses *indices (cluster) technique* [15] to prevent from infinite branching (possibly) caused by part restrictions.

Figure 1 presents the *choose-rule* which is augmented to add also labels, induced by part restrictions, and the new *M-rule* and *W-rule* which deal with part restrictions (we refer to them as *part rules*). All other rules— \sqcap -, \sqcup -, \exists -, \forall -, \forall_+ -, \geq - and \leq -rule—remain just as they are in [7]. \mathbf{T} is a completion tree for the examined concept D , and \mathcal{L} is a function, labelling the tree nodes with sets of concepts and tree edges with sets of roles, all occurring in D .

We also add the relevant *clashes with part restrictions*. For a node x , the set of labels $\mathcal{L}(x)$ is said to contain a clash (caused by *M-constructor*) iff for a concept C , a role S , and

- choose-rule:* If 1. $\odot S.C \in \mathcal{L}(x)$, x is not indirectly blocked, and
2. there is an S -neighbour y of x with $\{C, \sim C\} \cap \mathcal{L}(y) = \emptyset$
then $\mathcal{L}(y) \rightarrow \mathcal{L}(y) \cup \{E\}$ for some $E \in \{C, \sim C\}$.
- M-rule:* If 1. $MrS.C \in \mathcal{L}(x)$, x is not blocked, and no non-generating rule is applicable
to x and any of its ancestors, and
2. $\sharp S^{\mathbf{T}}(x, C) \leq r.\sharp S^{\mathbf{T}}(x)$ and $\sharp S^{\mathbf{T}}(x) < BAN_x$
then create a new successor y of x with $\mathcal{L}(\langle x, y \rangle) = \{S\}$, $\mathcal{L}(y) = \{C\}$,
and $y \neq z$, for any $z \in S^{\mathbf{T}}(x) \setminus \{y\}$.
- W-rule:* If 1. $WrS.C \in \mathcal{L}(x)$, x is not blocked, and no non-generating rule is applicable
to x and any of its ancestors, and
2. $\sharp S^{\mathbf{T}}(x, \sim C) > r.\sharp S^{\mathbf{T}}(x)$ and $\sharp S^{\mathbf{T}}(x) < BAN_x$
then create a new successor y of x with $\mathcal{L}(\langle x, y \rangle) = \{S\}$, $\mathcal{L}(y) = \{C\}$,
and $y \neq z$, for any $z \in S^{\mathbf{T}}(x) \setminus \{y\}$.

Figure 1: The new and modified completion rules

some $r \in \mathbf{Q}$ it holds: $MrS.C \in \mathcal{L}(x)$ and $\sharp S^{\mathbf{T}}(x, C) \leq r.\sharp S^{\mathbf{T}}(x)$ (cf. property 11. from the tableau definition). In the same way $\mathcal{L}(x)$ contains a clash (caused by W -constructor) iff $WrS.C \in \mathcal{L}(x)$ and $\sharp S^{\mathbf{T}}(x, \sim C) > r.\sharp S^{\mathbf{T}}(x)$.

The requirement in M -rule and W -rule no non-generating rule to be applicable ensures that 1. all concepts with part restrictions are already present in $\mathcal{L}(x)$ before the first application of part rules, and 2. if some non-generating rule becomes applicable after an M -rule or W -rule application, this rule will make all possible (re)labelling before the next application of a part rule. Both are necessary for the correct generation of successors, caused by part restrictions.

The checkup for reaching the *border amount of neighbours* of the current node x (denoted by BAN_x) is a kind of ‘horizontal blocking’ of the generation process, used to ensure inapplicability of part rules after a given moment. The notion is crucial for the generation process termination, and its use is based on the following result, which is the upshot of the indices technique. By the definition, a part restriction is *tree-satisfied*, when there is no clash with it.

Lemma 2. *Let all possible applications of \sqcap -rule and \sqcup -rule for the current node x be done. Then it can be calculated a natural number $BAN_x \geq 1$, having the following property: all part restrictions in $\mathcal{L}(x)$ which are simultaneously tree-satisfiable can be non-deterministically simultaneously tree-satisfied when the number of neighbours of x on any role at the uppermost level in some concept in $\mathcal{L}(x)$ becomes equal to BAN_x .*

5 Correctness of the Algorithm

As usual with tableaux algorithms we prove lemmas that the algorithm always terminates, and that it is sound and complete. The termination is ensured by pair-wise blocking, and by BAN -checkup, which guarantees finite (at most exponential—in case of usual binary coding of numbers) branching at a node. The build of a (possibly infinite) tableau from the finite completion tree follows the construction from [7].

Since the *internalization of terminologies* [1] is still possible in the presence of part restrictions following the technique presented in [7], we obtain finally:

Theorem 1. *The presented tableaux algorithm is a decision procedure for the satisfiability and subsumption of $ALCQPIH_{R^+}$ -concepts with respect to role hierarchies and terminologies.*

6 Conclusion

DL $\mathcal{ALCQPIH}_{R^+}$ augments \mathcal{ALCQIH}_{R^+} with the ability to express rational grading. We show that the decision procedure for the latter logic can naturally and easily be extended to capture the new one. This indicates once again that the approach which realizes rational grading independently from integer grading is fruitful, and can give in a convenient way rational grading extensions of known useful DLs.

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Dendral Resolution

Anton Zinoviev*

Faculty of Mathematics and Computer Science, Sofia University
E-mail: anton@lml.bas.bg

Abstract

In this paper the notion of “dendral resolution” is defined. This is an extremely restrictive but complete resolution refinement. Many important complete resolution refinements (such as lock resolution, positive hyperresolution and resolution with ordering) are special cases of dendral resolution.

1 Introduction to Dendral Resolution

Ordered resolution [3, section 6.6] is a very restrictive resolution refinement where the clauses are sequences of literals and resolution is permitted only on the first literal in each clause. This is a very simple and efficient method but it is not refutation complete. By using special treelike “dendral” clauses instead of sequences, a complete resolution refinement can be defined which is only a little less restrictive than ordered resolution. Maximum restriction is achieved when the initial dendral clauses approximate the clauses of ordered resolution.¹ It turns out, if we start with less restrictive initial dendral clauses, dendral resolution can be made equivalent to several important complete resolution refinements. This means that dendral resolution is one of the most restrictive complete resolution refinements.

Definition 1. (i) For any atomic formula λ , let $\bar{\lambda} = \neg\lambda$ and $\overline{\bar{\lambda}} = \lambda$.

(ii) We will fix a set Ξ whose elements will be called *indices*. *Disjunct* is an expression of the form $n: \lambda_1 \vee \dots \vee \lambda_k \vee \perp$ where $k \geq 0$, $n \in \Xi$ is an index and $\lambda_1, \dots, \lambda_k$ are literals (n is called *index* of this disjunct). *Dendral clause* is a finite rooted tree whose vertices are labelled with disjuncts. The intended semantics of a disjunct is the disjunction of its literals. The intended semantics of a dendral clause is the the disjunction of all literals of its disjuncts.

(iii) For any substitution s and a term or a formula τ , by $s(\tau)$ we will denote the result of the application of s to τ . If δ is a disjunct and \varkappa a dendral clause, by $s(\delta)$ and $s(\varkappa)$ we will denote the results of the application of s to all literals in δ and \varkappa , respectively.

(iv) The literal λ is an *instance* of μ if $\lambda = s(\mu)$ for some substitution s . The notions “instance of a disjunct” and “instance of a clause” are defined analogously.

(v) Two literals λ and μ are *variants* if λ is an instance of μ and μ is an instance of λ . Notice that λ and μ are variants if and only if λ can be obtained from μ by a bijective renaming of the variables. The notions “variant disjuncts” and “variant clauses” are defined analogously.

(vi) The substitution s is an *unifier* of a set Γ of literals, if $s(\lambda) = s(\mu)$ for any $\lambda, \mu \in \Gamma$.

(vii) The substitution s is an *unifier* of a set Γ of disjuncts, if all disjuncts in Γ have equal indices and s is an unifier of the sets $\Gamma_1, \Gamma_2, \Gamma_3, \dots$ where Γ_1 is the set of all literals last in a disjunct of Γ , Γ_2 is the set of all literals next to the last in a disjunct of Γ , and so on.

(viii) Given a set Γ of literals or disjuncts, the substitution s is its *most general unifier* if s is an unifier of Γ and for any unifier s' of Γ , there exists a substitution s'' , such that $s' = s'' \circ s$.

(ix) We will fix a partial function **mgu**, such that for any finite set Γ of literals or disjuncts, **mgu**(Γ) is a most general unifier of Γ , if there exists one, and **mgu**(Γ) is undefined, otherwise.

(x) $n: \mu_1 \vee \dots \vee \mu_k$ is a *shortening* of all disjuncts $n: \lambda_1 \vee \dots \vee \lambda_m \vee \mu_1 \vee \dots \vee \mu_k$, $m \geq 0$.

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¹In the terminology of Definition 1 (ii), this is so when the dendral clause contains only one disjunct.

Definition 9. For any set Γ of dendral clauses, let $\mathbf{res}^0(\Gamma)$ be the set of all normal forms of elements of Γ and $\mathbf{res}^{n+1}(\Gamma)$ be the union of $\mathbf{res}^n(\Gamma)$ and the set of all normal forms of activations of resolvents of variants of clauses belonging to $\mathbf{res}^n(\Gamma)$. Let $\mathbf{res}^\omega(\Gamma)$ be the union of all sets $\mathbf{res}^n(\Gamma)$. We say the clauses in $\mathbf{res}^\omega(\Gamma)$ are obtained from Γ by *dendral resolution*.

Definition 10. A dendral clause \varkappa is *coherent* if for any disjunct δ occurring in a node of \varkappa , for any disjunct ε occurring in a node successor (not necessarily immediate) of the node of δ and for any instances δ' and ε' of δ and ε , it is not true that $\varepsilon' \prec \delta'$.

Proposition. For any set Γ of coherent dendral clauses, all elements of $\mathbf{res}^\omega(\Gamma)$ are coherent.

Theorem for soundness and completeness. For any set Γ of coherent dendral clauses, \perp can be obtained from Γ by dendral resolution if and only if Γ is unsatisfiable.

Proof. (\Rightarrow) Since the transformations we use in order to obtain the normal form of a clause produce equivalent clauses, the normal form of a clause is equivalent to it. The activation of a clause follows from it. The variants of a clause are equivalent to it. The dendral resolvent of dendral clauses follows from them. Therefore, the elements of $\mathbf{res}^\omega(\Gamma)$ follow from Γ .

(\Leftarrow) See Lemma 4 in section 3. □

2 Dendral Resolution and Other Resolution Refinements

Linear resolution and its refinements do not fit directly into the framework of the standard resolution, because their definition, like that of tableau calculi with connection condition, relies on the form of the derivation [4, p. 155]. It turns out, however, that most other important complete resolution refinements are special cases of dendral resolution.

Lock resolution. [2] This is a resolution refinement where each literal is given an index. Then we require from the indices of the resolved literals to be maximal within their respective clauses. The factoring is permitted only for literals having equal indices.

The lock resolution can be modelled by dendral resolution. Instead of the lock-clause $(i_1)\lambda_1 \vee (i_2)\lambda_2 \vee \dots \vee (i_n)\lambda_n$, use the following dendral clause: $\perp \vee \{i_1: \lambda_1, i_2: \lambda_2, \dots, i_n: \lambda_n\}$. That is we use a separate disjunct for each literal. Then use the following coherent ordering: $n: \delta \preceq k: \varepsilon$ if and only if $n \leq k$.

Definition 8 ensures that the resolved literals will have maximal index and all literals participating in factoring will have equal indices.

Resolution with ordering. [6] This is a resolution refinement where we define an ordering of the literals (satisfying some requirements). Then we require from the resolved literals to be maximal with respect to this ordering within their respective clauses.

The resolution with ordering can be modelled by dendral resolution. Instead of the ordinary clause $\lambda_1 \vee \lambda_2 \vee \dots \vee \lambda_n$ use the following dendral clause: $\perp \vee \{1: \lambda_1, 1: \lambda_2, \dots, 1: \lambda_n\}$. That is we use equal indices everywhere and we give a separate disjunct to each literal. Then use the following coherent ordering: $1: \delta \preceq 1: \varepsilon$ if and only if $\delta = \varepsilon$ or both δ and ε are literals and δ precedes ε with respect the ordering of the literals.

Positive hyperresolution. [5] A clause is positive if it contains no negative literals. The positive hyperresolution is a resolution refinement where the clauses are ordered (that is they are sequences of literals). When we use a non-positive clause only the first literal among the

negative literals in the sequence is permitted to be used as a resolved literal. This implies that we are not permitted to produce resolvents from two non-positive clauses. The non-positive clause is called nucleus, the number of its negative literals is its valency and the positive clause is the electron. Each resolvent has smaller valency than the nucleus, so if the initial valency is n , after n resolvents we will obtain a positive clause, called hyperresolvent.

The positive hyperresolution can be modelled by dendral resolution. Instead of the ordinary clause $\lambda_1 \vee \lambda_2 \vee \dots \vee \lambda_n \vee \mu_1 \vee \mu_2 \vee \dots \vee \mu_m$, where $\lambda_1, \dots, \lambda_n$ are negative literals and μ_1, \dots, μ_m are positive, we use $\perp \vee \{1: \lambda_1 \vee \dots \vee \lambda_n, 1: \mu_1, \dots, 1: \mu_m\}$. Then we use the following coherent ordering: $1: \delta \preceq 1: \varepsilon$ if and only if $\delta = \varepsilon$ or δ contains only positive literals and ε contains a negative literal.

It is not difficult to generalise this construction in order to see that the semantic resolution and the semantic hyperresolution can be modelled by dendral resolution as well.

Positive hyperresolution with ordering. This is a positive hyperresolution where we have an ordering of the positive literals. We require from the resolved literal in a positive clause to be maximal within the clause.

In order to model the positive hyperresolution with ordering by dendral resolution we use the same dendral clauses as before but we change the definition of the coherent ordering: $1: \delta \preceq 1: \varepsilon$ if and only if $\delta = \varepsilon$ or δ contains only positive literals and ε contains a negative literal or both δ and ε are positive literals and δ precedes ε with respect the ordering of the literals.

3 Proof of the Theorem for Soundness and Completeness

Lemma 1. *Let Γ be an unsatisfiable finite set of ground coherent dendral clauses, such that no two disjuncts occurring in elements of Γ have equal indices. Suppose there exists a strict total ordering $<$ of the indices, such that for any ground disjuncts $n: \delta$ and $k: \varepsilon$, $n: \delta < k: \varepsilon$ is equivalent to $n < k$. Then $\perp \in \mathbf{res}^\omega(\Gamma)$.*

Proof. We are going to prove this lemma using a variation of the method proposed in [1].

By induction on the total number of the literals occurring in elements of Γ . Without loss of generality we may assume that all elements of Γ are normal.

Among all disjuncts occurring in clauses of Γ , let δ be the one with the smallest index. Let n be the index of δ and λ be the last literal in δ . Remove λ in order to obtain Γ' . For any $\varepsilon' \in \mathbf{res}^\omega(\Gamma')$ there is a corresponding $\varepsilon \in \mathbf{res}^\omega(\Gamma)$, such that ε can be obtained from ε' by adding λ at the end of each disjunct with index n and adding at various places in ε' new disjuncts $n: \lambda$. By induction hypothesis, $\perp \in \mathbf{res}^\omega(\Gamma')$. Therefore, $\mathbf{res}^\omega(\Gamma)$ has an element of the form $\perp \vee \{n: \lambda, n: \lambda, \dots, n: \lambda\}$. The clause $n: \lambda$ is an activation of this element.

Remove from Γ an arbitrary occurrence of the literal $\bar{\lambda}$ in order to obtain Γ'' . Since we can use resolvents with the clause $n: \lambda$ in order to remove any occurrence of $\bar{\lambda}$ as first literal in the root of a clause, for any $\varepsilon'' \in \mathbf{res}^\omega(\Gamma'')$ there is a corresponding $\varepsilon \in \mathbf{res}^\omega(\Gamma)$, such that ε can be obtained from ε'' by adding a few literals $\bar{\lambda}$ to it. By induction hypothesis, $\perp \in \mathbf{res}^\omega(\Gamma'')$. Therefore, $\mathbf{res}^\omega(\Gamma)$ has an element whose only literals are $\bar{\lambda}$, so we can use resolvents with $n: \lambda$ in order to obtain \perp . \square

Lemma 2. *Let Γ be an unsatisfiable finite set of ground coherent dendral clauses, such that no two disjuncts occurring in elements of Γ have equal indices. Then $\perp \in \mathbf{res}^\omega(\Gamma)$.*

Proof. First, some definitions. *Wide disjunct* is a pair $\langle n: \langle \lambda_i \rangle_i \rangle$ of index n and infinite sequence of literals. Any disjunct $n: \lambda_k \vee \lambda_{k-1} \vee \dots \vee \lambda_1$ is called a *shortening* of this wide disjunct.

Consider the following binary relation between wide disjuncts δ and ε : let $\mathbf{p}(\delta, \varepsilon)$ be true if $\delta = \varepsilon$ or for some shortenings δ' of δ and ε' of ε , $\delta' \prec \varepsilon'$. From Definition 6 (ii) we can conclude that \mathbf{p} is a partial ordering. Let \mathbf{q} be an arbitrary total ordering extending \mathbf{p} .

For any disjuncts δ occurring in an element of Γ , let $\hat{\delta}$ be an arbitrarily chosen wide disjunct, such that δ is a shortening of $\hat{\delta}$. Define a strict total ordering $<$ over indices: let $n < k$ if and only if $n \neq k$ and there exist disjuncts δ and ε occurring in Γ whose indices are n and k , so that $\mathbf{q}(\hat{\delta}, \hat{\varepsilon})$ is true. Let \preceq' be the minimal partial ordering over disjuncts, such that for any ground disjuncts $n: \delta$ and $k: \varepsilon$, $n: \delta \prec' k: \varepsilon$ is equivalent to $n < k$. This is a coherent ordering.

All disjuncts occurring in a clause of $\mathbf{res}^\omega(\Gamma)$ are shortenings of disjuncts occurring in a clause of Γ . Therefore, for any disjuncts δ and ε occurring in a clause of $\mathbf{res}^\omega(\Gamma)$, $\delta \prec \varepsilon$ implies $\delta \prec' \varepsilon$. Consequently, if some clause follows from Γ by dendral resolution using \preceq' , then it will follow by dendral resolution using \preceq (the opposite is not necessarily true).

According to Lemma 1, if we use the ordering \preceq' instead of \preceq , then \perp will follow from Γ by dendral resolution. Therefore, $\perp \in \mathbf{res}^\omega(\Gamma)$. \square

Lemma 3. *Let Γ be an unsatisfiable finite set of ground coherent clauses. Then $\perp \in \mathbf{res}^\omega(\Gamma)$.*

Proof. Let Ξ' be the Cartesian product of the set of the indices Ξ and the set of the natural numbers. Replace any index n of a disjunct occurring in a clause of Γ with $\langle n, k \rangle \in \Xi'$ in order to obtain a set Γ' . While doing this, chose the natural numbers k in such a way, that no two indices in Γ' are equal.

Let \preceq' be the partial ordering, such that for any two disjuncts with indices from Ξ' ,

$$\langle n_1, k_1 \rangle: \lambda_1 \vee \dots \vee \lambda_m \prec' \langle n_2, k_2 \rangle: \mu_1 \vee \dots \vee \mu_l$$

is true if and only if $n_1: \lambda_1 \vee \dots \vee \lambda_m \prec n_2: \mu_1 \vee \dots \vee \mu_l$ is true.

Since \preceq' is a coherent ordering, from Lemma 2 it follows that \perp is derivable from Γ by dendral resolution with indices from Ξ' and coherent ordering \preceq' . If a clause is derivable from Γ by dendral resolution with indices from Ξ' and coherent ordering \preceq' , then a corresponding clause where the indices $\langle n, k \rangle$ are replaced with n is derivable from Γ by dendral resolution with indices from Ξ and coherent ordering \preceq . Therefore, $\perp \in \mathbf{res}^\omega(\Gamma)$. \square

Lemma 4. *Let Γ be an unsatisfiable set of coherent clauses. Then $\perp \in \mathbf{res}^\omega(\Gamma)$.*

Proof. Theorem of Herbrand implies that there is a finite set Γ' of ground instances of elements of Γ , such that Γ' is unsatisfiable. Lemma 3 implies that $\perp \in \mathbf{res}^\omega(\Gamma')$. Now, Definition 6 (i) permits us to use the usual lifting argument in order to conclude that $\perp \in \mathbf{res}^\omega(\Gamma)$. \square

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Part III
Posters

List of Posters

An Independent, Equivalent Axiomatic System for MV-Algebras

*İbrahim Şentürk and Tahsin Öner (Ege University, Turkey),
Gülşah Öner (Dokuz Eylül University, Turkey)*

Recent proof mining results for PDE theory and fixed point theory and other ongoing applications
of proof theory to analysis

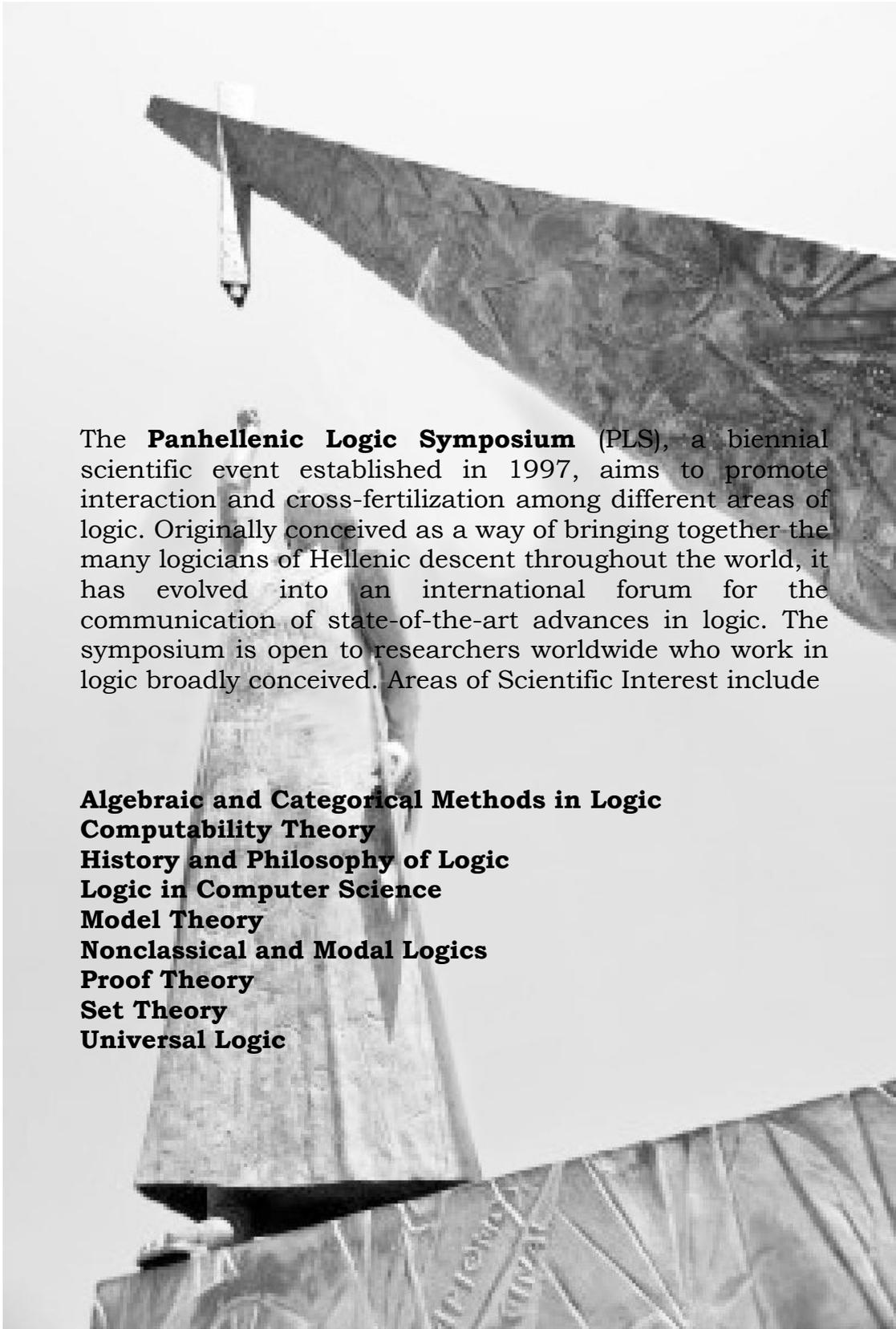
Angeliki Koutsoukou-Argyraki (Technische Universität Darmstadt)

Comparing C and Zilber's Exponential Fields: Zero Sets of Exponential Polynomials

Giuseppina Terzo (Seconda Università degli Studi di Napoli, Italy)

Logical Generalized Continued Fractions

Ilya Makarov (National Research University Higher School of Economics, Russia)



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